ANALYTICAL CALCULATION OF QUASINORMAL MODES FOR NON-ROTATING KALUZA-KLEIN BLACK HOLE WITH SQUASHED HORIZONS

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Abstract: We study the Kaluza-Klein black holes, with squashed horizons. The system is perturbed by a massless scalar field. We analytically calculate the solution to the wave equation, which are called the quasinormal modes. We use an approximation technique for the non-rotating case, where the frequencies to solution are a set of discrete complex number. Our analytical results agree with the numerical results, where the difference percentage is 0.64% for the lowest fundamental mode $\lambda = 0$. However as $\lambda$ increasing, this percentage becomes larger.

Introduction: The extra dimensions have become mandatory to sustain or generalize the principles or theories in many physics areas of research. The Kaluza-Klein theory has combined the general relativity and electrodynamics into a 5-dimensional theory. Black holes contained in the general relativity can be viewed as singularities that dictate the spacetime curvature and the behaviors of particles and fields in the systems. The fields in the black hole systems must satisfy the boundary conditions that only ingoing wave at the horizons of the black holes and only outgoing wave or the vanishing wave at the infinity are allowed. The waves that obey these boundary conditions are called quasinormal modes and their associated frequencies as quasinormal frequencies.1 The gravitational wave signals from supermassive black holes are expected to be detected by Laser Interfermeter Space Antenna (LISA).2 In this work, the non-rotating Kaluza-Klein black holes with the squashed horizon are studied and the quasinormal modes of these systems are analytically calculated, where we expect the frequencies to be a set of discrete complex number.

In this introduction section we describe the Kaluza-Klein black hole system and the squashed shape of the horizon. The wave equation in this curved spacetime is considered.

The five-dimensional rotating Kaluza-Klein black holes with the squashed horizons have the metric in the form of

$$ds^2 = -dt^2 + \frac{\sum \Delta k^2 dr^2 + r^2 + a^2}{4}[k(\sigma_1^2 + \sigma_2^2) + \sigma_3^2] + \frac{\mu}{r^2 + a^2}(dt - \frac{a}{2})^2$$

(1)

where

$$\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi,$$

$$\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi,$$

$$\sigma_3 = d\psi + \cos \theta d\phi$$

(2)

$$\Sigma(r) = r^2(r^2 + a^2),$$

$$\Delta(r) = (r^2 + a^2)^2 - \mu^2$$

(3)

$$k(r) = \frac{(r_c^2 - r^2)(r^2 - r^2)}{(r^2 - r^2)^2}.$$  

(4)

Parameters $\mu$ and $a$ correspond to black hole mass and angular momentum respectively. The variables have the range $0 \leq r \leq r_c$, $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$, where the radius
has the upper limit \( r_\infty \). The sphere of the horizon is squashed by the term \( k(r) \). The singularities of the metric are obtained by setting \( \Delta(r_\infty) = 0 \). \( r_+ \) and \( r_- \) are outer and inner horizons respectively, which are depended on \( \mu \) and \( a \) as

\[
r_\pm = \sqrt{\left(\mu - 2a^2\right) \pm \sqrt{\mu^2 - 4a^2 \mu^2}}
\]

(6)

Since the metric is apparently singular at \( r = r_\infty \), the radial coordinate \( r \) is restricted within the range \( 0 < r < r_\infty \). Note that, if we take \( r_\infty \to \infty \) which causes \( k(r) \to 1 \) therefore our metric reduces to five-dimensional Kerr black hole. Moreover, the shape of the event horizon is also characterized by the function \( k(r_+) \).

The metric above describes the space-time geometry of the rotating squashed Kaluza-Klein black hole which looks like a five-dimensional squashed black hole near the horizons, and like the Kaluza-Klein geometry at \( r \to r_\infty \). To see the asymptotic behavior of this metric, let us define a new radial coordinate

\[
\rho = \rho_0 \frac{r^2}{r_\infty^2 - r^2}
\]

(7)

where

\[
\rho_0 = \sqrt{\frac{k_0(r_\infty^2 + a^2)}{4}}, \text{ and } k_0 = k(r = 0) = \frac{(r_\infty^2 + a^2)^2 - \mu r_\infty^2}{r_\infty^4}
\]

(8)

In a new radial coordinate \( \rho \) varies from 0 to \( \infty \) while \( r \) varies from 0 to \( r_\infty \). By transforming (1) via a new coordinate (7) and taking limit \( \rho \to \infty \), the metric becomes

\[
d s^2 = -d t^2 + d \rho^2 + \rho^2 (\sigma_1^2 + \sigma_2^2) + \frac{r_\infty^2 + a^2}{4} \sigma_3^2 + \frac{\mu}{r^2 + a^2} \left( d t - \frac{a}{2} \sigma_3 \right)^2
\]

(9)

To remove the cross-term between \( d t \) and \( \sigma_3 \), let define new coordinates as

\[
\tilde{\psi} = \psi - \frac{2\mu a}{(r_\infty^2 + a^2)^2 + \mu a^2} t
\]

\[
\tilde{t} = \sqrt{\frac{(r_\infty^2 + a^2)^2 - \mu a^2}{(r_\infty^2 + a^2)^4 + \mu a^2 t}}
\]

(10)

and define a new notation \( \tilde{\sigma}_3 = d \tilde{\psi} + \cos \theta d \phi \), and replace all these new coordinates and notation into (9). Then, the asymptotic structure of the rotating squashed Kaluza-Klein black hole is revealed

\[
d s^2 = -d \tilde{t}^2 + d \rho^2 + \rho^2 (\sigma_1^2 + \sigma_2^2) + \frac{(r_\infty^2 + a^2)^2}{4(r_\infty^2 + a^2)} \tilde{\sigma}_3^2
\]

(11)

The first three terms on the RHS of (11) represent a four dimensional Minkowski spacetime while the rest is a twisted \( S^1 \) bundle. The size of the compactified dimension at infinity is also obtained

\[
r_\infty^2 = \frac{(r_\infty^2 + a^2)^2 + \mu a^2}{r_\infty^2 + a^2}
\]

(12)

The size of the extra dimension depends on three parameters \( r_\infty, \mu \) and \( a \). Note that, for \( a \to 0 \) or \( r_\infty^2 \gg a^2 \), the radius of the compactified dimension (12) could be interpreted by \( r_\infty \).

\textbf{Klein-Gordon equation in curved background}
Our aim is to study a scalar field which evolves in the rotating squashed black hole spacetime. Hence, we need to construct equation of motion for a scalar field in curved background. An equation of motion for a real scalar field is so called Klein-Gordon equation. To derive a Klein-Gordon equation in a curved background, let consider an action for a single scalar field in curved spacetime

\[ S = \int \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} (\nabla_{\mu} \Phi)(\nabla_{\nu} \Phi) - \frac{1}{2} m^2 \Phi^2 \right] d^4x \]  

(13)

Here \( m \) stands for mass of a scalar field, where \( g \) is determinant of the metric tensor. For a scalar field case, it is possible to replace the covariant derivative with an ordinary partial derivative. By varying this action with respect to the scalar field, we obtain an equation of motion for a scalar field in curved background.

\[ \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi \right) - m^2 \Phi = 0 \]  

(15)

This equation will be used as an important part of our calculation in the next section. Its solution describes the behavior of the scalar field in the curved spacetime. For simplicity in our work, we consider only for a massless scalar field.

**Analytical determination of approximate quasinormal frequencies:** In this section we study the wave equation of the perturbed scalar field. The equation is separable. We analytically solve the radius wave equation. We obtain and present the frequency-constrain equation which corresponds to the each quasinormal mode.

**Equation of motion for a real scalar field in a rotating squashed Kaluza-Klein black hole**

We are going to calculate an equation of motion for a scalar particle in our particular metric (1). First, we have defined the proper time \( dt = B d\tau \) and \( B = \frac{(r_c^2 + a^2)^2}{2 \rho r^3} \) is a constant.\(^5\)

The former metric (1) becomes

\[ ds^2 = -B^2 d\tau^2 + \frac{\Delta}{\Sigma} k^2 dr^2 + \frac{r^2 + a^2}{4} \left[ k(\sigma_1^2 + \sigma_2^2) + \sigma_3^2 \right] + \frac{\mu}{r^2 + a^2} (Bd\tau - \frac{a}{2} \sigma_1)^2 \]

(16)

Therefore, we can calculate components of metric tensor \( g_{\mu\nu} \), and its inverse \( g^{\mu\nu} \) as

\[
g_{\mu\nu} = \begin{pmatrix}
    \left(1 - \frac{\mu}{r^2 + a^2}\right) B^2 & 0 & 0 & -\frac{a\mu\cos\theta}{2(r^3 + a^3)} B \\
    0 & \frac{\Sigma k^2}{\Delta} & 0 & 0 \\
    0 & 0 & \frac{k^2(r^2 + a^2)}{4} & 0 \\
    -\frac{a\mu\cos\theta}{2(r^3 + a^3)} B & 0 & 0 & \frac{r^2 + a^2}{4} (k \sin^2 \theta + \cos^2 \theta) + \frac{\mu a^2 \cos^2 \theta}{4(r^3 + a^3)} \\
    -\frac{a\mu}{2(r^3 + a^3)} B & 0 & 0 & \frac{\cos \theta \left( (r^2 + a^2)^2 + \mu a^2 \right)}{4(r^3 + a^3)} \\
    -\frac{a\mu}{2(r^3 + a^3)} B & 0 & 0 & \frac{(r^2 + a^2)^2 + a^2 \mu}{4(r^3 + a^3)}
\end{pmatrix}
\]
where \( \sqrt{-g} = \frac{k^2 \sin \theta B}{8 \sqrt{(r^2 + a^2)\Sigma}} \), In our calculation, we denote spacetime indices by \((\tau, r, \theta, \phi, \psi) \rightarrow (0, 1, 2, 3, 4)\). It is convenient to use a new radial coordinate \( \rho \) which defined by (7). Then, take the ansatz for a scalar field \( \Phi(\tau, \rho, \theta, \phi, \psi) = e^{-i\omega \tau} R(\rho) e^{im\phi} S(\theta) \), where \( S(\theta) \) is the spheroidal harmonics. After changing to a new radial coordinate and put the ansatz into (15), we can separate the angular variable part from the radial and the time parts. For the angular part, it reads

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d}{d\theta} \right] S(\theta) - \left[ \frac{(m - \lambda \cos \theta)^2}{\sin^2 \theta} - E_{l\lambda} \right] S(\theta) = 0 \tag{17}
\]

Here the eigenvalue of the angular equation is \( E_{l\lambda} = l(l+1) - \lambda^2 \). The parameters \( l \) is the angular number of the variable \( \theta \) and \( \lambda \) is integer number for the 5th dimension represented by the angle variable, \( \psi \). However, the time variable can be removed in the final step of the calculation. So, the only remaining part is the radial component which takes the form

\[
\Theta \frac{d^2 R(\rho)}{d\rho^2} + \frac{d\Theta}{d\rho} \frac{dR(\rho)}{d\rho} + \left[ \tilde{N}^2 + \Lambda - l(l+1) + \tilde{\lambda}^2 \right] R(\rho) = 0 \tag{18}
\]

and

\[
\Theta(\rho) = \frac{(r_+^2 + a^2)}{4r_+^2 \rho_0^2} \left[ (\rho r_+^2 + a^2 (\rho + \rho_0))^2 - \mu \rho (\rho + \rho_0) r_+^2 \right],
\]

\[
\tilde{N}^2 = \frac{\mu \rho_0^2 (\rho + \rho_0)^4}{N^2 (r_+^2 + a^2)^4} \left[ \omega - \frac{\lambda a N^2 (r_+^2 + a^2)}{r_+^4 \rho_0^3} \right]^2,
\]

\[
\Lambda = \frac{4 \rho_0^2 r_+^6 (\rho + \rho_0)^2 \omega^2}{N^2 (r_+^2 + a^2)^4} - \frac{4 \lambda^2 (\rho + \rho_0)^2}{r_+^4 + a^2},
\]

\[
N^2 = \frac{\rho + \rho_0}{\rho^2 + a^2 \rho_0},
\]

\[
\rho_+ = \rho_0 \frac{r_+^2}{r_+^2 - r_0^2},
\]

\[
\rho_0 = \frac{\rho^2}{r_+^2 - r_-^2},
\]

\[
\rho_0 = \frac{(\mu - 2a^2)\pm \sqrt{\mu^2 - 4a^2 \mu}}{2}.
\]
In order to obtain the quasinormal frequencies $\omega$, we have to solve (18) under certain boundary conditions as mentioned before. In this work, we limit ourselves to the case of non-rotation $a = 0$ and the equation of motion (18) becomes

$$\rho (\rho - \rho_c) \frac{d^2 R(\rho)}{d \rho^2} + (\rho - \rho_c)(2 \rho - \rho_c) \frac{d R(\rho)}{d \rho} + \mu \omega^2 \rho (\rho + \rho_o) R(\rho)$$

$$+ \left( \rho - \rho_c \right) \left[ \frac{4 \rho_o^2 (\rho + \rho_o) \rho \omega^2}{r_w^2} - \frac{4 \lambda^2 (\rho + \rho_o)^2}{r_w^2} - E_{\text{inj}} \right] R(\rho) = 0$$

(20)

To simplify the solution let us separate the singularity at horizon by writing $R(\rho)$ as,

$$R(\rho) = (\rho - \rho_c)^\alpha F(\rho)$$

where

$$\alpha = -\frac{i \omega r_w}{2 \left( \frac{r_w^2}{\mu} - 1 \right)}$$

(21)

The minus sign of $\alpha$ represents the incoming wave at the horizon. After substituting $R(\rho)$ in equation (20) the wave equation changes to

$$\rho (\rho - \rho_c) \frac{d^2 F(\rho)}{d \rho^2} + 2 \alpha \rho \frac{d F(\rho)}{d \rho} + (2 \rho - \rho_c) \frac{d F(\rho)}{d \rho} + (\alpha (\alpha - 1) + 2 \alpha) F(\rho) +$$

$$\frac{\mu \omega^2}{r_w^2} \left[ (\rho - \rho_c)^2 + 2 (\rho - \rho_c)(\rho_c + \rho_o) + \rho_c (\rho - \rho_c) + 2 \rho_c (\rho_c + \rho_o) + (\rho_c + \rho_o)^2 \right] F(\rho) +$$

$$\left[ \frac{4 \rho_o^2 (\rho_c + \rho_o) \rho \omega^2}{r_w^2} - \frac{4 \lambda^2 (\rho + \rho_o)^2}{r_w^2} - E_{\text{inj}} \right] F(\rho) = 0$$

(22)

Let’s define a new variable $x = \frac{\rho}{\rho_c}$, where $x = 1$ at the horizon. The radius wave equation is reduced to

$$x (1 - x) \frac{d^2 F(x)}{d x^2} + \left[ 1 - (2 \alpha + 2) x \right] \frac{d F(x)}{d x} + \left[ \alpha' + \beta' x + \gamma' x^2 \right] F(x) = 0$$

(23)

$$\alpha' = \frac{4 \lambda^2 \rho_o^2}{r_w^2} + E_{\text{inj}} - \alpha$$

$$\beta' = -\frac{\mu \omega^2}{r_w^2} \rho_c (\rho_c + \rho_o) - \rho_o \rho_c \omega^2 + \frac{8 \lambda^2 \rho_c \rho_o}{r_w^2}$$

(24)

$$\gamma' = -\omega^2 \rho_c^3 + \frac{4 \lambda^2 \rho_c^2}{r_w^2}$$

To further simplify the solution let us define $F = e^{C x} H(x)$, where we choose $C^2 = -\gamma'$. Then the above equation, (23) changes to

$$x (x - 1) \frac{d^2 H}{d x^2} + \left[ 2 C x (x - 1) - 1 + 2 (1 + \alpha) x \right] \frac{d H}{d x} + (\alpha' - C) H + \left[ -C^2 + 2 C + 2 C \alpha + \beta' \right] x H = 0$$

(25)

Change the variable from $x$ to $v = x - 1$, where at the horizon $v = 0$. Next divide the equation with $-2 C x$ and the wave equation becomes
\[
(-2Cv) \frac{d^2 H}{d(-2Cv)^2} + \left[1 + 2\alpha - (-2Cv) + 1 \frac{1}{x} \right] \frac{dH}{d(-2Cv)} - \left[1 + \alpha + \frac{\beta^\prime + \gamma^\prime}{2C} \right] H - \frac{\alpha^\prime - C}{2Cx} H = 0
\]

(26)

The wave equation, near the horizon, \( x=1 \), can be approximated as

\[
(-2Cv) \frac{d^2 H}{d(-2Cv)^2} + \left[1 + 2\alpha - (-2Cv) \right] \frac{dH}{d(-2Cv)} - \left[2 \frac{1}{2} + \alpha + \frac{\alpha^\prime + \beta^\prime + \gamma^\prime}{2C} \right] H = 0
\]

(27)

The solutions to the above wave equation, (27) are the confluent hypergeometric functions. However we need only the ingoing solution to the horizon in order to satisfy the boundary condition

\[ R = e^{x \left( \rho - \rho_{+} \right)} A \left( \hat{\alpha}; \hat{\beta}; -2Cv \right) \]

(28)

where \( \hat{\alpha} = \frac{1}{2} + \alpha + \frac{\alpha^\prime + \beta^\prime + \gamma^\prime}{2C} \) and \( \hat{\beta} = 1 + 2\alpha \). In the region, \( x \to \infty \) the solution must be vanished in order to satisfy the other boundary condition where the potential in this region diverges. We can approximate the solution at the infinity by using the property of the confluent hypergeometric function

\[
R(v \to \infty) \approx e^{Cv-i\pi \hat{\alpha}} (-2C)^{-\hat{\alpha}} v^{\hat{\alpha}-\hat{\beta}} \frac{1}{\Gamma(\hat{\beta} - \hat{\alpha})} + e^{-Cv} (-2C)^{\hat{\beta}-\hat{\alpha}} v^{\hat{\alpha}+\hat{\beta}} \frac{1}{\Gamma(\hat{\alpha})}
\]

(29)

As \( v \to \infty \), the first term decays, while the second term diverges. To get rid of the divergence, set the argument of the Gamma function to be negative integer.

\[ \hat{\alpha} = \frac{1}{2} + \alpha + \frac{\alpha^\prime + \beta^\prime + \gamma^\prime}{2C} = -n, \quad n = 0,1,2,3,... \]

(30)

The above equation, (30), is a constrain equation for the frequencies, where it can be written down in term of a dimensionless \( \omega \rho \), parameter as

\[
\left(2n + 1 - 2\lambda \rho_{+} \right) \left(\frac{\omega \rho}{\mu}\right) \left[1 - \frac{\lambda^2}{\mu} \right] = 0
\]

(31)

The quasinormal frequencies can be solved from (31). Hence, this equation is the fourth order polynomial, then there are four roots to equation (31). After giving the proper numerical value to each parameter, equation (31) can be solved algebraically. We have found that there is only one root that gives positive real number and negative imaginary number, corresponding to the ingoing wave at the horizon and the decaying wave at the far away region respectively. To emphasize, these results are only the approximate solution of quasinormal frequencies because we have used approximation technique to reduce the difficulty of solving wave equation. We will compare our result with numerical work in the next section

Results and Discussion of the quasinormal modes and frequencies: The quasinormal frequencies are obtained from equation (31). We put some specific parameters in order to compare with the numerical result as the following

Table 1 The quasinormal frequencies from WKB method and our analytical result, where \( l=10, \rho_{0}/\rho_{+} = 3 \) or \( r_{+}/r_{-} = 2 \) and \( n = 2 \)

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>WKB (^6)</th>
<th>Analytical work</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.5149 - 0.15956i</td>
<td>3.4921 - 0.16667i</td>
</tr>
<tr>
<td>0.5</td>
<td>3.5336 - 0.15821i</td>
<td>3.4940 - 0.16643i</td>
</tr>
<tr>
<td></td>
<td>3.5898 - 0.15413i</td>
<td>3.4999 - 0.16572i</td>
</tr>
<tr>
<td>---</td>
<td>-----------------</td>
<td>------------------</td>
</tr>
<tr>
<td>1</td>
<td>3.6842 - 0.14729i</td>
<td>3.5098 - 0.16455i</td>
</tr>
<tr>
<td>1.5</td>
<td>3.8178 - 0.13759i</td>
<td>3.5236 - 0.16291i</td>
</tr>
<tr>
<td>2</td>
<td>3.9924 - 0.12493i</td>
<td>3.5413 - 0.16083i</td>
</tr>
<tr>
<td>2.5</td>
<td>4.2103 - 0.10912i</td>
<td>3.5629 - 0.15832i</td>
</tr>
<tr>
<td>3</td>
<td>4.4747 - 0.08988i</td>
<td>3.5883 - 0.15539i</td>
</tr>
</tbody>
</table>

From both results the imaginary part is negative, causing the wave vanishing in the far away region. As $\lambda$ increases, the real part increases while the imaginary part decreases.

Our frequencies change more slowly than WKB result when $\lambda$ increases. To improve our frequency result, we can take our approximated solution as the zero order perturbation. We can continue to perform the first order. Also we can add the rotation (black hole, $a \neq 0$) to the problem in our future work.

References:

Acknowledgements: This work is partially supported by Thailand Research Fund and partially supported by Srinakharinwirot University research fund and the Office of the Higher Education Commission. S.P. also partially supported by 90th year Chulalongkorn Scholarship and Thailand Toray Science Foundation. We would like to thank Asst. Prof. Dr. Piyabut Burikham for the valuable comments.

Keywords: squashed black holes, Kaluza-Klein theory, quasinormal modes, quasinormal frequencies