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Contributed Paper

Convergence Properties of pth Order Diagonally Implicit Block Backward Differentiation Formulas

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Abstract

This paper investigates the convergence properties for diagonally implicit 2-point block backward differentiation formulas of order two, three and four. The formulation of the method is reviewed from the literature. The order of the method is verified. The concepts of consistency and zero stability are considered to prove the convergence of the method.

Keywords order, diagonally implicit, block method, consistency, zero stability, convergence

1. Introduction

In general, numerical methods are converge if the numerical solutions approach the exact solutions. Butcher [1] described that the convergence refers to the ability of the method to approximate the solution of differential equations to any required accuracy, if sufficiently many small step size are taken. Some researchers have proposed their strategies to determine the convergence of linear multistep method (LMM) in the family of block method. For instance, the technique to determine the order, consistency, stability and convergence of 3-point block method is described by Ehigie et al. [2]. Recently, Abualnaja [3] stated the order, error constant and convergence of block procedure for k-step LMM, where $k=1,\ 2,\ 3$. In the context of block backward differentiation formulas (BBDF), Ibrahim et al. [4] presented the consistency and zero stability of the fully implicit 2-point BBDF of order three to show that the method converges. Meanwhile, Nasir et al. [5] discussed the zero stability of the fully implicit 2-point BBDF of order five. This motivation leads us to study on the convergence properties of the diagonally implicit 2-point block backward differentiation formulas with various order.

2. Review of the Method

In this section, we will review the formulation of diagonally implicit 2-point block backward differentiation formulas for order two (DI2BBDF(2)), order three (DI2BBDF(3)) and order four (DI2BBDF(4)) given by Zawawi et al. [6]. The derivation for each formula involves different interpolating points to compute the approximated solutions at y_{n+1} and y_{n+2} concurrently. To obtain the diagonally implicit characteristic, the first point of formula must has one interpolating point less than the second point of formula. The formulas are given as follows:

DI2BBDF(2):

$$y_{n+1}+\frac{1}{3}y_{n-1}-\frac{4}{3}y_n=\frac{2}{3}hf_{n+1},$$

$$y_{n+2}-\frac{2}{11}y_{n-1}+\frac{9}{11}y_n-\frac{18}{11}y_{n+1}=\frac{6}{11}hf_{n+2}. \quad (1)$$

DI2BBDF(3):

$$y_{n+1}-\frac{2}{11}y_{n-2}+\frac{9}{11}y_{n-1}-\frac{18}{11}y_n=\frac{6}{11}hf_{n+1},$$

$$y_{n+2}+\frac{3}{25}y_{n-2}-\frac{16}{25}y_{n-1}+\frac{36}{25}y_n-\frac{48}{25}y_{n+1}=\frac{12}{25}hf_{n+2}. \quad (2)$$

DI2BBDF(4):

$$y_{n+1}+\frac{3}{25}y_{n-3}-\frac{16}{25}y_{n-2}+\frac{36}{25}y_{n-1}-\frac{48}{25}y_n=\frac{12}{25}hf_{n+1},\ y_{n+2}-\frac{12}{137}y_{n-3}+\frac{75}{137}y_{n-2}-\frac{200}{137}y_{n-1}+\frac{300}{137}y_n-\frac{300}{137}y_{n+1}=\frac{60}{137}hf_{n+2}. \quad (3)$$

The derived formulas can be expressed in the general form of LMM

$$\sum_{j=0}^{k}\alpha_j y_{n+j-1}=h\sum_{j=0}^{k}\beta_j f_{n+j-1} \quad (4)$$

where α_j and β_j are matrices coefficients and k is defined as the order of the implicit method. It has to be noted that the order of the diagonally implicit method is $k-1$. For easy reference, the coefficients of formulas (1), (2) and (3) are tabulated in the following tables respectively.

Table 1. Coefficients of DI2BBDF(2).

α_j	α_0	α_1	α_2	α_3	β_2	β_3
First point	$\frac{1}{3}$	$-\frac{4}{3}$	1	0	$\frac{2}{3}$	0
Second point	$-\frac{2}{11}$	$\frac{9}{11}$	$-\frac{18}{11}$	1	0	$\frac{6}{11}$

Table 2. Coefficients of DI2BBDF(3).

α_j	α_0	α_1	α_2	α_3	α_4	β_3	β_4
First point	$-\frac{2}{11}$	$\frac{9}{11}$	$-\frac{18}{11}$	1	0	$\frac{6}{11}$	0
Second point	$\frac{3}{25}$	$-\frac{16}{25}$	$\frac{36}{25}$	$-\frac{48}{25}$	1	0	$\frac{12}{25}$

Table 3. Coefficients of DI2BBDF(4).

α_j	α_0	α_1	α_2	α_3	α_4	α_5	β_4	β_5
First point	$\frac{3}{25}$	$-\frac{16}{25}$	$\frac{36}{25}$	$-\frac{48}{25}$	1	0	$\frac{12}{25}$	0
Second point	$-\frac{12}{137}$	$\frac{75}{137}$	$-\frac{200}{137}$	$\frac{300}{137}$	$\frac{300}{137}$	1	0	$\frac{60}{137}$

3. ORDER OF THE METHOD

In this section, the order of the proposed method will be verified. The definition as given by Lambert [7] is applied to determine the order of DI2BBDF(2), DI2BBDF(3) and DI2BBDF(4).

Definition 1

The LMM is said to be of order p if $C_0 = C_1 = \ldots = C_p = 0,\ C_{p+1} \neq 0$ and the general form of constant C_q is defined as follows:

$$C_q = \sum_{j=0}^{k} \left(\frac{1}{q!} j^q \alpha_j - \frac{1}{(q-1)!} j^{q-1} \beta_j\right), q = 0,\ 1,\ 2, \ldots \quad (5)$$

Order of DI2BBDF(2)

Substitute all coefficients α_j and β_j from Table 1 into (5), the order of the method is determined by computing $C_0,\ C_1,\ C_2 \cdots$ until the Definition 1 is satisfied.

$$C_0 = \sum_{j=0}^{3} \left(\frac{1}{0!} j^0 \alpha_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \sum_{j=0}^{3} \left(\frac{1}{1!} j^1 \alpha_j - \frac{1}{0!} j^0 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \sum_{j=0}^{3} \left(\frac{1}{2!} j^2 \alpha_j - \frac{1}{1!} j^1 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_3 = \sum_{j=0}^{3} \left(\frac{1}{3!} j^3 \alpha_j - \frac{1}{2!} j^2 \beta_j\right) = \begin{bmatrix} -\frac{2}{9} \\ 0 \end{bmatrix}.$$

Since we obtained $C_0 = C_1 = C_2 = 0$ and $C_3 \neq 0$, the DI2BBDF(2) is order 2.

Order of DI2BBDF(3)

Applying the same procedure, the order is determined as follows:

$$C_0 = \sum_{j=0}^{4} \left(\frac{1}{0!} j^0 \alpha_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \sum_{j=0}^{4} \left(\frac{1}{1!} j^1 \alpha_j - \frac{1}{0!} j^0 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \sum_{j=0}^{4} \left(\frac{1}{2!} j^2 \alpha_j - \frac{1}{1!} j^1 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_3 = \sum_{j=0}^{4} \left(\frac{1}{3!} j^3 \alpha_j - \frac{1}{2!} j^2 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_4 = \sum_{j=0}^{4} \left(\frac{1}{4!} j^4 \alpha_j - \frac{1}{3!} j^3 \beta_j\right) = \begin{bmatrix} -\frac{3}{22} \\ 0 \end{bmatrix}.$$

It is observed that $C_0 = C_1 = C_2 = C_3 = 0$ and $C_4 \neq 0$. This is verified that the DI2BBDF(3) is order 3.

Order of DI2BBDF(4)

Similarly, the order of DI2BBDF(4) is determined as follows:

$$C_0 = \sum_{j=0}^{5} \left(\frac{1}{0!} j^0 \alpha_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \sum_{j=0}^{5} \left(\frac{1}{1!} j^1 \alpha_j - \frac{1}{0!} j^0 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \sum_{j=0}^{5} \left(\frac{1}{2!} j^2 \alpha_j - \frac{1}{1!} j^1 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_3 = \sum_{j=0}^{5} \left(\frac{1}{3!} j^3 \alpha_j - \frac{1}{2!} j^2 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_4 = \sum_{j=0}^{5} \left(\frac{1}{4!} j^4 \alpha_j - \frac{1}{3!} j^3 \beta_j\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_5 = \sum_{j=0}^{5} \left(\frac{1}{5!} j^5 \alpha_j - \frac{1}{4!} j^4 \beta_j\right) = \begin{bmatrix} \frac{12}{25} \\ 0 \end{bmatrix}.$$

We obtained $C_0 = C_1 = C_2 = C_3 = C_4 = 0$. Since $C_5 \neq 0$, we conclude that the method is order 4.

4. CONVERGENCE PROPERTIES

The definition of convergence is given by Lambert [7] as follows:

Definition 2

The necessary conditions for LMM to be convergent are that it be consistent and zero-stable. This section describes the consistency and zero stability of the proposed methods by analyzing their coefficients, α and β .

4.1 Consistency

The methods can be proved to be consistent if and only if the following conditions are satisfied:

$$\sum_{j=0}^{k}\alpha_j = 0,\ \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k}\beta_j. \qquad (6)$$

Consistency of DI2BBDF(2)

We satisfy the conditions (6) by substituting all coefficients in Table 1 as follows

$$\sum_{j=0}^{3}\alpha_j = \alpha_0+\alpha_1+\alpha_2+\alpha_3 = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\sum_{j=0}^{3} j\alpha_j = (0)\alpha_0+(1)\alpha_1+(2)\alpha_2+(3)\alpha_3 = \begin{bmatrix}\frac{2}{3}\\ \frac{6}{11}\end{bmatrix},$$

$$\sum_{j=0}^{3}\beta_j = \beta_0+\beta_1+\beta_2+\beta_3 = \begin{bmatrix}\frac{2}{3}\\ \frac{6}{11}\end{bmatrix}.$$

Therefore, we can conclude that the DI2BBDF(2) is consistent.

Consistency of DI2BBDF(3)

The similar approach is applied by substituting all coefficients in Table 2 into (6) as follows

$$\sum_{j=0}^{4}\alpha_j = \alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4 = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\sum_{j=0}^{4} j\alpha_j = (0)\alpha_0+(1)\alpha_1+(2)\alpha_2+(3)\alpha_3+(4)\alpha_4 = \begin{bmatrix}\frac{6}{11}\\ \frac{12}{25}\end{bmatrix},$$

$$\sum_{j=0}^{4}\beta_j = \beta_0+\beta_1+\beta_2+\beta_3+\beta_4 = \begin{bmatrix}\frac{6}{11}\\ \frac{12}{25}\end{bmatrix}.$$

We have shown the conditions (6) are satisfied. Hence the DI2BBDF(3) is consistent.

Consistency of DI2BBDF(4)

The consistency of the DI2BBDF(4) is determined as follows

$$\sum_{j=0}^{5}\alpha_j = \alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\sum_{j=0}^{5} j\alpha_j = (0)\alpha_0+(1)\alpha_1+(2)\alpha_2+(3)\alpha_3+(4)\alpha_4+(5)\alpha_5 = \begin{bmatrix}\frac{12}{25}\\ \frac{60}{137}\end{bmatrix},$$

$$\sum_{j=0}^{5}\beta_j = \beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5 = \begin{bmatrix}\frac{12}{25}\\ \frac{60}{137}\end{bmatrix}.$$

Thus, the DI2BBDF(4) is consistent.

4.2 Zero Stability

It has to be mentioned that the technique to determine the zero stability of LMM has been presented by Ibrahim et al. [9]. In this section, we will determine the zero stability of DI2BBDF(2), DI2BBDF(3) and DI2BBDF(4) based on the following definition.

Definition 3

LMM is said to be zero-stable if the roots $R_j, j = 1(1)k$ of the first characteristic polynomial $\rho(R) = \det[\sum_{i=0}^{k} A_i R^{k-i}] = 0, A_0 = -I$, satisfies $|R_j| \le 1$. If one of the roots is +1, we call this root the principal root of $\rho(R)$.

Fatunla [8] stated that the stability properties of the method are determined through the application to standard linear test equation

$$y' = \lambda y,\ \lambda < 0, \qquad (7)$$

Zero-stable of DI2BBDF(2)

Substitute equation (7) into (1) will produce

$$y_{n+1} - \frac{2}{3}\lambda h y_{n+1} = -\frac{1}{3}y_{n-1} + \frac{4}{3}y_n,$$

$$y_{n+2} - \frac{6}{11}\lambda h y_{n+2} = \frac{2}{11}y_{n-1} - \frac{9}{11}y_n + \frac{18}{11}y_{n+1}. \qquad (8)$$

Let $\hat{h} = \lambda h$,equations (8) are transformed in the matrix form

$$\begin{bmatrix}1-\frac{2}{3}\lambda h & 0\\ -\frac{18}{11} & 1-\frac{6}{11}\lambda h\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\end{bmatrix} = \begin{bmatrix}-\frac{1}{3} & \frac{4}{3}\\ \frac{2}{11} & -\frac{9}{11}\end{bmatrix}\begin{bmatrix}y_{n-1}\\ y_n\end{bmatrix}.$$

where

$$A=\begin{bmatrix}1-\frac{2}{3}\hat{h} & 0\\ -\frac{18}{11} & 1-\frac{6}{11}\hat{h}\end{bmatrix},\ B=\begin{bmatrix}-\frac{1}{3} & \frac{4}{3}\\ \frac{2}{11} & -\frac{9}{11}\end{bmatrix}.$$

Solve $\det(At-B)$, the first characteristic polynomial, $\rho(t,\hat{h})$ is given by

$$\rho(t,\hat{h})=t^2-\frac{40}{33}t^2\hat{h}-\frac{34}{33}t+\frac{4}{11}t^2\hat{h}^2-\frac{8}{11}t\hat{h}+\frac{1}{33}.$$

Consider $\hat{h}=0$ and solve the stability polynomial to obtain $t_1=0.030303$ and $t_2=1$. Hence, DI2BBDF(2) is zero-stable. Since one of the roots is +1, we call this root as principal root.

Zero-stable of DI2BBDF(3)

Substitute (7) into (2) to obtain

$$y_{n+1}-\frac{6}{11}\lambda hf_{n+1}=\frac{2}{11}y_{n-2}-\frac{9}{11}y_{n-1}+\frac{18}{11}y_n$$

$$y_{n+2}-\frac{12}{25}\lambda hf_{n+2}=-\frac{3}{25}y_{n-2}$$

$$+\frac{16}{25}y_{n-1}-\frac{36}{25}y_n+\frac{48}{25}y_{n+1}. \quad (9)$$

Let $\hat{h}=\lambda h$, convert (9) into the matrix form

$$\begin{bmatrix}1-\frac{6}{11}\lambda h & 0\\ -\frac{48}{25} & 1-\frac{12}{25}\lambda h\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\end{bmatrix}=$$

$$\begin{bmatrix}0 & \frac{2}{11}\\ -\frac{3}{25} & \frac{6}{25}\end{bmatrix}\begin{bmatrix}y_{n-3}\\ y_{n-2}\end{bmatrix}+\begin{bmatrix}-\frac{9}{11} & \frac{18}{11}\\ -\frac{3}{25} & -\frac{3}{25}\end{bmatrix}\begin{bmatrix}y_{n-1}\\ y_n\end{bmatrix},$$

where

$$A=\begin{bmatrix}1-\frac{6}{11}\hat{h} & 0\\ -\frac{48}{25} & 1-\frac{12}{25}\hat{h}\end{bmatrix},\ B=\begin{bmatrix}-\frac{9}{11} & \frac{18}{11}\\ \frac{16}{25} & -\frac{36}{25}\end{bmatrix},$$

$$C=\begin{bmatrix}0 & \frac{2}{11}\\ 0 & -\frac{3}{25}\end{bmatrix}.$$

By solving $\rho(t,\hat{h})=\det(At^2-Bt-C)$, the first characteristic polynomial is given by

$$\rho(t,\hat{h})=-\frac{1}{55}t-\frac{27}{275}t^2-\frac{18}{275}\hat{h}t^2-\frac{243}{275}t^3$$

$$-\frac{324}{275}\hat{h}t^3+t^4-\frac{282}{275}\hat{h}t^4+\frac{72}{275}\hat{h}^2t^4.$$

Let $\hat{h}=0$, we compute the following roots

$$t_1=-0.0581818-0.121642i,-0.0581818$$
$$+0.121642i,\ t_2=0,\ t_3=1.$$

Therefore, the DI2BBDF(3) is zero-stable method based on the Definition 3.

Zero-stable of DI2BBDF(4)

Substitute (7) into (3) to produce

$$y_{n+1}-\frac{12}{25}hf_{n+1}=-\frac{3}{25}y_{n-3}+\frac{16}{25}y_{n-2}$$
$$-\frac{36}{25}y_{n-1}+\frac{48}{25}y_n,$$

$$y_{n+2}-\frac{60}{137}hf_{n+2}=\frac{12}{137}y_{n-3}-\frac{75}{137}y_{n-2}$$
$$+\frac{200}{137}y_{n-1}-\frac{300}{137}y_n+\frac{300}{137}y_{n+1}. \quad (10)$$

Let $\hat{h}=\lambda h$, rearrange (10) in the matrix form

$$\begin{bmatrix}1-\frac{12}{25}\lambda h & 0\\ -\frac{300}{137} & 1-\frac{60}{137}\lambda h\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\end{bmatrix}=$$

$$\begin{bmatrix}-\frac{3}{25} & \frac{16}{25}\\ \frac{12}{137} & -\frac{75}{137}\end{bmatrix}\begin{bmatrix}y_{n-3}\\ y_{n-2}\end{bmatrix}+\begin{bmatrix}-\frac{36}{25} & \frac{48}{25}\\ \frac{200}{137} & -\frac{300}{137}\end{bmatrix}\begin{bmatrix}y_{n-1}\\ y_n\end{bmatrix}.$$

where.

$$A=\begin{bmatrix}1-\frac{12}{25}\hat{h} & 0\\ -\frac{300}{137} & 1-\frac{60}{137}\hat{h}\end{bmatrix},\ B=\begin{bmatrix}-\frac{36}{25} & \frac{48}{25}\\ \frac{200}{137} & -\frac{300}{137}\end{bmatrix},$$

$$C=\begin{bmatrix}-\frac{3}{25} & \frac{16}{25}\\ \frac{12}{137} & -\frac{75}{137}\end{bmatrix}.$$

The first characteristic polynomial, $\rho(t,\hat{h})$ is given by

$$\rho(t,\hat{h}) = \frac{33}{3425} - \frac{176}{3425}t - \frac{1314}{3425}t^2 - \frac{216}{685}\hat{h}t^2 - \frac{1968}{3425}t^3 - \frac{1152}{685}\hat{h}t^3 + t^4 - \frac{3144}{3425}\hat{h}t^4 + \frac{144}{685}\hat{h}^2t^4$$

Hence, we have the roots of the stability polynomial

$$t_1 = -0.101305,\ t_2 = 1,\ t_3 = 0.263353 - 0.160483i,\ t_4 = -0.263353 + 0.160483i.$$

Since the roots satisfy the Definition 3, it can be concluded that the DI2BBDF(4) is zero-stable.

5. Conclusions

In this paper, the necessary conditions for the convergence of diagonally implicit 2-point block backward differentiation formulas of order two, three and four are presented. Furthermore, the order of the method is verified. The diagonally implicit method also satisfied the conditions of consistency and zero stability. Hence, the method is proven to be convergent.

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