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# The *k*-nacci Sequences and The Generalized Order-*k* Pell Sequences in The Semi-Direct Product of Finite Cyclic Groups

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## ABSTRACT

Let $p$ be a prime and $G = \left\langle y, x \mid y^{2^p-1} = e, x^p = e, x^{-1}yx = y^r \right\rangle$ be a semi-direct product $B$ by $A$ where $B$ and $A$ both cyclic groups of order $2^p-1$ and $p$, respectively. This paper discusses the periods of the $k$-nacci sequences and the generalized order-$k$ Pell sequences in the group $G$ for $r = 2$ and $p \geq 3$.

**Keywords:** Period, $k$-nacci sequence, generalized order-$k$ Pell sequence, cyclic group, semi-direct product

## 1. INTRODUCTION

As it is well-known Fibonacci numbers and related numbers appear in modern research in many fields from Nature and Architecture to theoretical physics; see for example, [3,6,7,14]. The study of Fibonacci sequences in groups began with the earlier work of Wall [17]. Knox examined the $k$-nacci ($k$-step Fibonacci) sequences in the finite groups [15]. Karaduman and Aydin studied the periods of the 2-step general Fibonacci sequences in dihedral groups $D_n$ [11]. Lü and Wang contributed to study of the Wall number for the $k$-step Fibonacci sequence [16]. Kilic and Tasci studied the $k$ sequences of the generalized order-$k$ Pell numbers [13]. Deveci and Karaduman studied the generalized order-$k$ Pell sequences modulo $m$ and the generalized order-$k$ Pell sequences in a finite groups and obtained the periods of the generalized order-$k$ Pell sequences in the dihedral group $D_n$ [5]. The Pell sequence, the generalized order-$k$ Pell sequence and their properties have been studied by several authors; see for example, [2,8-10, 12].

Let $f_n^{(k)}$ denote the $n$th member of the $k$-step Fibonacci sequence defined as

$$f_n^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \quad \text{for } n > k \tag{1}$$

with boundary conditions $f_i^{(k)} = 0$ for $1 \leq i < k$ and $f_k^{(k)} = 1$. Reducing this sequence by a modulus $m$, we can get a repeating sequence, which we denote by

$$f(k,m) = \left(f_1^{(k,m)}, f_2^{(k,m)}, \ldots, f_n^{(k,m)}, \ldots\right),$$

where $f_n^{(k,m)} = f_n^{(k)}$ (mod $m$). We then have that $\left(f_1^{(k,m)}, f_2^{(k,m)}, \ldots, f_k^{(k,m)}\right) = (0,0, \ldots\ 0,1)$ and it has the same recurrence relation as in (1).
For more information see [16].

**Theorem 1.1** $f(k, m)$ is a periodic sequence [16].
Let $h_k(m)$ denote the smallest period of $f(k, m)$, called the period of $f(k, m)$ or the Wall number of the $k$-step Fibonacci sequence modulo $m$.
For more information see [16].

**Definition 1.1** Let $h_{k\,(a_1, a_2, \ldots, a_k)}(m)$ denote the smallest period of the integer-valued recurrence relation $u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-k}$, $u_1 = a_1, u_2 = a_2, \cdots, u_k = a_k$ when each entry is reduced modulo $m$ [4].

**Lemma 1.1** For $a_0, a_1, \cdots, a_{k-1}, b_0, b_1, \cdots, b_{k-1}, m \in \mathbb{Z}$ with $m > 0$, $a_0, a_1, \cdots, a_{k-1}$ not all congruent to zero modulo $m$ and $b_0, b_1, \cdots, b_{k-1}$ not all congruent to zero modulo $m$,

$$h_{k\,(a_1, a_2, \ldots, a_k)}(m) = h_{k\,(b_1, b_2, \ldots, b_k)}(m) \quad [4].$$

**Definition 1.2** A $k$-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, \cdots, x_n, \cdots$ for which, given an initial (seed) set $x_0, \cdots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases}$$

We also require that the initial elements of the sequence, $x_0, \cdots, x_{j-1}$, generate the group, thus forcing the $k$-nacci sequence to reflect the structure of the group. The $k$-nacci sequence of a group generated by $x_0, \cdots, x_{j-1}$ is denoted by $F_k, (G; x_0, \cdots, x_{j-1})$ [15].

It is well-known that a sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

**Theorem 1.2** A $k$-nacci sequence in finite group is simply periodic [15].

In [15], Knox had denoted the period of the sequence $F_k, (G; x_0, \cdots, x_{j-1})$ by $P_k, (G; x_0, \cdots, x_{j-1})$.

In [13], Kilic and Tasci defined the $k$ sequences of the generalized order-$k$ Pell numbers as follows:

for $n > 0$ and $1 \le i \le k$

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i, \tag{2}$$

with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \le n \le 0,$$

where $P_n^i$ is the $n$th term of the $i$th sequence. If $k = 2$, the generalized order-$k$ Pell sequence, $\{P_n^k\}$, is reduced to the usual Pell sequence, $\{P_n\}$.
When $i = k$ in (1), we call $P_n^k$ the generalized $k$- Pell number.
In [13], the generalized order-$k$ Pell matrix $R$ had been given as

$$R=\left[r_{ij}\right]_{k\times k}=\begin{bmatrix}2 & 1 & \cdots & 1 & 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}.$$

Reducing the generalized order-$k$ Pell sequence by a modulus $m$, we can get a repeating sequence, denoted by

$$\left\{P^{k,m}\right\}=\left\{P^{k,m}_{1-k},P^{k,m}_{2-k},\cdots,P^{k,m}_{0},P^{k,m}_{1},P^{k,m}_{2},\cdots,P^{k,m}_{n},\cdots\right\},$$

where $P^{k,m}_{n}=P^{k}_{n}$ (mod $m$). It has the same recurrence relation as in (2).
For more information see [5].

**Theorem 1.3** $\{P^{k,m}\}$ is a periodic sequence [5].
Let the notation $hP_k(m)$ denotes the smallest period of $\{P^{k,m}\}$, called the period of the generalized order-$k$ Pell sequence modulo $m$. When $k = 2$, $hP_2(m)$ is the period of the Pell sequence modulo $m$.
For more information see [5].

**Definition 1.3** A generalized order-$k$ Pell sequence in a finite group is a sequence of group elements $x_0, x_1,\cdots, x_n,\cdots$ for which, given an initial (seed) set $x_0, \cdots, x_{j-1}$, each element is defined by

$$x_n=\begin{cases}x_0x_1\cdots\left(x_{n-1}\right)^2 & \text{for } j\le n<k,\\ x_{n-k}x_{n-k+1}\cdots\left(x_{n-1}\right)^2 & \text{for } n\ge k.\end{cases}$$

It is require that the initial elements of the sequence, $x_0,\cdots, x_{j-1}$, generate the group, thus, forcing the generalized order-$k$ Pell sequence to reflect the structure of the group. We denote the generalized order-$k$ Pell sequence of a group $G$ generated by $x_0,\cdots, x_{j-1}$ by $Q_k$, $(G; x_0, \cdots, x_{j-1})$. We call a generalized order-2 Pell sequence of group elements a Pell sequence of a finite group [5].

**Theorem 1.4** A generalized order-$k$ Pell sequence in a finite group is periodic [5].
In [5], Deveci and Karaduman had denoted the period of the sequence $Q_k$ $(G; x_0, \cdots, x_{j-1})$ by $PerQ_k$ $(G; x_0, \cdots, x_{j-1})$.

From the definitions it is clear that the periods of both the $k$-nacci sequence and the generalized order-$k$ Pell sequence in a group depend on the chosen generating set and order in which the assignments of $x_0, x_1, x_2,\cdots, x_{j-1}$ are made.

**Lemma 1.2** Suppose that $P_B=\langle \mathrm{y}\mid \mathrm{s}\rangle$ and $P_A=\langle \mathrm{x}\mid \mathrm{r}\rangle$ are presentations for the groups $B$ and $A$, respectively under the maps

$$y \mapsto k_y \in B \text{ and } x \mapsto a_x \in A.$$

Then we have a presentation

$$P = \langle \mathrm{y}, \mathrm{x} \mid \mathrm{s}, \mathrm{r}, \mathrm{t} \rangle,$$

for $G = B \times_\theta A$, where $\mathrm{t} = \left\{ yx\lambda_{\mathrm{yx}}^{-1}x^{-1} \mid y \in \mathrm{y},\ x \in \mathrm{x} \right\}$ and $\lambda_{yx}$ is a word on $y$ representing the element $(k_y)\theta_{a_x}$ of $B$ ($a \in A$, $k \in B$, $x \in \mathrm{x}$, $y \in \mathrm{y}$ [1].

Let $B$ be a cyclic group of order $n$ ($n \in$ □) with a presentation $P_B = \langle y \mid y^n \rangle$, and let $A$ be a cyclic group of order $p$ ($p$ is a prime) with a presentation $P_A = \langle x \mid x^p \rangle$. Then, by Lemma 1.2, a presentation for $G = B \times_\theta A$ is given by

$$P = \langle y, x \mid y^n = e, x^p = e, x^{-1}yx = y^r \rangle, \tag{3}$$

where

(i) $(r, n) = 1$,

(ii) $(r-1), nt) = t$ with $t = (r-1, n)$,

(iii) $r^p \equiv 1 \pmod{nt}$ for $r, t \in$ □ [1].

Now let us take $r = 2$ and $n = 2^p - 1$ in conditions (i), (ii) and (iii). ( So that $t = 1$ in (ii) and (iii)). Then, by substituting these values in (3), we get

$$P_G = \langle y, x \mid y^{2^p-1} = e, x^p = e, x^{-1}yx = y^2 \rangle,$$

as a presentation for the group $G$ [1].

This paper discusses the periods of the $k$-nacci sequences and the generalized order-$k$ Pell sequences in the group $P_G$ for $r = 2$ and $p \geq 3$.

## 2. The *k*-nacci Sequences in The Group $P_G$

**Lemma 2.1** Let $t \in$ □ and $P$ be a prime such that $p \geq 3$.

**i.** The elements $(h_2(p))$th and $+1)$th of the 2-nacci sequences $F_2(P_G; y, x)$ and $F_2(P_G; x, y)$ are in the following forms, respectively:

$$x_{h_2(p)} = y,\ x_{h_2(p)+1} = xy^a$$

and

$$x_{h_2(p)} = xy^a,\ x_{h_2(p)+1} = y,$$

where $a$ is an integer such that $0 \leq a < 2^p - 1$.

Also, the elements $(t.h_2(p))$th and $(t.h_2(p)+1)$th of the 2-nacci sequences $F_2(P_G; y, x)$ and $F_2(P_G; x, y)$ are as follows, respectively:

$$x_{h_2(p)} = y,\ x_{h_2(p)+1} = xy^b$$

and

$$x_{h_2(p)} = xy^b,\ x_{h_2(p)+1} = y,$$

where $b$ is an integer such that $ta \equiv b \pmod{2^p - 1}$.

**ii.** The elements $(\alpha)$th of the $k$-nacci sequences $F_k(P_G; y, x)$ and $F_k(P_G; x, y)$ for all $t.h_k(p) - (k-2) \leq a \leq t.h_k(p)+1$ are in the following forms, respectively:

$$x_{t.h_k(p)-(k-2)} = y^{a_1}, x_{t.h_k(p)-(k-2)+1} = y^{a_2}, \cdots, x_{t.h_k(p)-1} = y^{a_{k-2}}, x_{t.h_k(p)} = y^{a_{k-1}}, x_{t.h_k(p)+1} = xy^{a_k},$$

where $a_1, a_2, ..., a_k$ are integers such that $0 \le a_1, a_2, ..., a_k < 2^p - 1$ and

$$x_{t.h_k(p)-(k-2)} = y^{b_1}, x_{t.h_k(p)-(k-2)+1} = y^{b_2}, \cdots, x_{t.h_k(p)-1} = y^{b_{k-2}}, x_{t.h_k(p)} = xy^{b_{k-1}}, x_{t.h_k(p)+1} = y^{b_k},$$

where $b_1, b_2, ..., b_k$ are integers such that $0 \le b_1, b_2, ..., b_k < 2^p - 1$.

**Proof.** We prove this by direct calculation. We first note that in the group defined by the presentation $\langle y, x \mid y^{2^p-1} = e, x^p = e, x^{-1}yx = y^2 \rangle$, $|y| = 2^p-1$, $|x| = p$ and $yx = xy^2$.
**i.** If $k = 2$, consider the recurrence relations defined by the following:

$f_n$ denote the nth member of the Fibonacci sequence

$$f_n = f_{n-2} + f_{n-1} \text{ for } n \ge 3, \text{ where } f_1 = 0,\ f_2 = 1;$$

$$v_n = 2^{f_{n-1}}.v_{n-2} + v_{n-1} \text{ for } n \ge 3, \text{ where } v_1 = 1,\ v_2 = 0$$

and

$$u_n = u_{n-2} + u_{n-1} \text{ for } n \ge 3, \text{ where } u_1 = 1,\ u_2 = 0;$$

$$w_n = 2^{u_{n-1}}.w_{n-2} + w_{n-1} \text{ for } n \ge 3, \text{ where } w_1 = 0,\ w_2 = 1.$$

Then the 2-nacci sequences $F_2(P_G; y, x)$ and $F_2(P_G; x, y)$ are in the following forms, respectively:

$$x_0 = y, x_1 = x, x_2 = yx = xy^2, \cdots, x_{h_2(p)} = x^{f_{h_2(p)+1}} y^{v_{h_2(p)+1}}, x_{h_2(p)+1} = x^{f_{h_2(p)+2}} y^{v_{h_2(p)+2}}, \cdots,$$
$$x_{(t.h_2(p))} = x^{f_{(t.h_2(p))+1}} y^{v_{(t.h_2(p))+1}}, x_{(t.h_2(p)+1)} = x^{f_{(t.h_2(p))+2}} y^{v_{(t.h_2(p))+2}}, \cdots,$$

and

$$x_0 = x, x_1 = y, x_2 = xy, \cdots, x_{h_2(p)} = x^{u_{h_2(p)+1}} y^{w_{h_2(p)+1}}, x_{h_2(p)+1} = x^{u_{h_2(p)+2}} y^{w_{h_2(p)+2}}, \cdots,$$
$$x_{(t.h_2(p))} = x^{u_{(t.h_2(p))+1}} y^{w_{(t.h_2(p))+1}}, x_{(t.h_2(p)+1)} = x^{u_{(t.h_2(p))+2}} y^{w_{(t.h_2(p))+2}}, \cdots,$$

using Lemma 1.1 the elements $(h_2(p))$th and $(h_2(p)+1)$th of the 2-nacci sequences $F_2(P_G; y, x)$ and $F_2(P_G; x, y)$ are obtanied as follows, respectively:

$$x_{h_2(p)} = y,\ x_{h_2(p)+1} = xy^a$$

and

$$x_{h_2(p)} = xy^a,\ x_{h_2(p)+1} = y,$$

where $a$ is an integer such that $v_{h_2(p)+2} \equiv a(\text{mod } 2^p - 1)$ and $w_{h_2(p)+1} \equiv a(\text{mod } 2^p - 1)$.
Also, it is easy to see from Lemma 1.1 that the elements $(t.h_2(p))$th and $(t.h_2(p)+1)$th of the 2-nacci sequences $F_2(P_G; y, x)$ and $F_2(P_G; x, y)$ are as follows, respectively:

$$x_{h_2(p)} = y,\ x_{h_2(p)+1} = xy^{t.a}$$

and

$$x_{h_2(p)} = xy^{t.a},\ x_{h_2(p)+1} = y.$$

If $t.a \equiv b \pmod{2^p - 1}$, we get the elements $(t.h_2(p))$th and $(t.h_2(p)+1)$th of the 2-nacci sequences $F_2(P_G; y, x)$ and $F_2(P_G; x, y)$, respectively

$$x_{h_2(p)} = xy^b,\ x_{h_2(p)+1} = y$$

and

$$x_{h_2(p)} = y,\ x_{h_2(p)+1} = xy^b.$$

**ii.** If $k \geq 3$, consider the recurrence relations defined by the following:

$u_n = u_{n-k} + u_{n-(k-1)} +,\cdots+ u_{n-1}$ for $n \geq k+1$, where $u_1 = 0$, $u_2 = 1$ and $u_i = 2^{i-3}$ for $3 \leq i \leq k$

and

$v_n = v_{n-k} + v_{n-(k-1)} +,\cdots+ v_{n-1}$ for $n \geq k+1$, where $v_1 = 1$, $v_2 = 0$ and $u_i = 2^{i-3}$ for $3 \leq i \leq k$

Then the $k$-nacci sequences $F_k(P_G; y, x)$ and $F_k(P_G; x, y)$ are in the following forms, respectively:

$$x_0 = y,\ x_1 = x,\ x_2 = yx = xy^2,\ x_3 = x^2y^6, \cdots, x_k = x^{2^{k-2}}y^{\beta}, \cdots,$$

$$x_{(t.h_k(p))-(k-2)} = x^{u_{(t.h_k(p))-(k-2)+1}}y^{a_1},\ x_{(t.h_k(p))-(k-2)+1} = x^{u_{(t.h_k(p))-(k-2)+2}}y^{a_2}, \cdots,$$

$$x_{(t.h_k(p))} = x^{u_{(t.h_k(p))+1}}y^{a_{k-1}},\ x_{(t.h_k(p))+1} = x^{u_{(t.h_k(p))+2}}y^{a_k}, \cdots,$$

where $\beta, a_1, a_2, \ldots, a_k$ are integers such that $0 \leq \beta, a_1, a_2, \ldots, a_k < 2^p - 1$ and

$$x_0 = x,\ x_1 = y,\ x_2 = xy,\ x_3 = x^2y^3, \cdots, x_k = x^{2^{k-2}}y^{\lambda}, \cdots,$$

$$x_{(t.h_k(p))-(k-2)} = x^{v_{(t.h_k(p))-(k-2)+1}}y^{b_1},\ x_{(t.h_k(p))-(k-2)+1} = x^{v_{(t.h_k(p))-(k-2)+2}}y^{b_2}, \cdots,$$

$$x_{(t.h_k(p))} = x^{v_{(t.h_k(p))+1}}y^{b_{k-1}},\ x_{(t.h_k(p))+1} = x^{v_{(t.h_k(p))+2}}y^{b_k}, \cdots,$$

where $\lambda, b_1, b_2, \ldots, b_k$ are integers such that $0 \leq \lambda, b_1, b_2, \ldots, b_k < 2^p - 1$.

Using Lemma 1.1 the elements $(\alpha)$th of the $k$-nacci sequences $F_k(P_G; y, x)$ and $F_k(P_G; x, y)$ for all $t.h_k(p) - (k-2) \leq \alpha \leq t.h_k(p)+1$ are obtained as follows, respectively:

$$x_{(t.h_k(p))-(k-2)} = y^{a_1},\ x_{(t.h_k(p))-(k-2)+1} = y^{a_2}, \cdots, x_{(t.h_k(p))} = y^{a_{k-1}},\ x_{(t.h_k(p))+1} = xy^{a_k}, \cdots,$$

and

$$x_{(t.h_k(p))-(k-2)} = y^{b_1},\ x_{(t.h_k(p))-(k-2)+1} = y^{b_2}, \cdots, x_{(t.h_k(p))} = xy^{b_{k-1}},\ x_{(t.h_k(p))+1} = y^{b_k}, \cdots.$$

Lemma 2.1 gives immediately

**Theorem 2.1** Let $p$ be a prime such that $p \geq 3$.

**i.** If the elements $(h_2(p))$th and $(h_2(p)+1)$th of the 2-nacci sequence $F_2(P_G; y, x)$ are $x_{h_2} =$ y and $x_{h_2(p)+1} = xy^a$, respectively, then $P_2(P_G; y, x)$ and $P_2(P_G; x, y) = \eta.h_2(p)$, where $\eta$ is the last natural number such that $\eta.a \equiv 0 \pmod{2^p - 1}$.

**ii.** The $k$-nacci sequences $F_k(P_G; y, x)$ and $F_k(P_G; x, y)$ are from layers of length $h_k(p)$.

## 3. The Generalized order-$k$ Pell Sequences in The Group $P_G$

**Definition 3.1** Let $hP_{k(a_0, a_1,\dots, a_{k-1})}(m)$ denotes the smallest period of the integer-valued recurrence relation $u_n = 2u_{n-1}+u_{n-2}+\dots+u_{n-k}$, $u_0 = a_0$, $u_1 = a_1,\dots, u_{k-1} = a_{k-1}$ when each entry is reduced modulo $m$.

**Lemma 3.1** For $a_0, a_1,\dots, a_{k-1}, b_0, b_1,\dots, b_{k-1}, m\in$ with $m>0$, $a_0, a_1,\dots, a_{k-1}$ not all congruent to zero modulo $m$ and $b_0, b_1,\dots, b_{k-1}$ not all congruent to zero modulo $m$,

$$hP_{k(a_0, a_1,\dots, a_{k-1})}(m) = hP_{k(b_0, b_1,\dots, b_{k-1})}(m).$$

**Proof.** Let $S_k = \{(s_0, s_1,\dots, s_{k-1}) \mid 0 \le s_i \le m-1\}$. Then we have $|S_k| = m^k$ being finite, that is there are $m^k$ $k$-tuples of the elements of $S^k$. Thus, we have the integers $l$ and $t$ such that $0 \le l \le m^k$, $0 \le t \le k-1$ and

$$x_0 = a_0, x_1 = a_1, \cdots, x_{k-1} = a_{k-1}, \cdots, x_{l.k+t} = b_0, x_{x_{l.k+t+1}} = b_1, \cdots, x_{l.k+t+k-1} = b_{k-1}, \cdots.$$

So, we get $hP_{k(a_0, a_1,\dots, a_{k-1})}(m) = hP_{k(b_0, b_1,\dots, b_{k-1})}(m)$.

It is easy to see that $P_G = D_3$ for $P= 2$. For the generalized order-$k$ Pell sequences in the dihedral group $D_n$ see [5].

**Lemma 3.2** Let $t\in$ and $p$ be a prime such that $p \ge 3$.

**i.** The elements $(hP_2(p))$th and $(hP_2(p)+1)$th of the Pell sequences $Q_2(P_G; y, x)$ and $Q_2(P_G; x, y)$ are in the following forms, respectively:

$$x_{hP_2(p)} = y,\ x_{hP_2(p)+1} = xy^c$$

and

$$x_{hP_2(p)} = xy^c,\ x_{hP_2(p)+1} = y,$$

where $c$ is an integer such that $0 \le c < 2^p - 1$.

Also, the elements $(t.hP_2(p))$th and $(t.hP_2(p)+1)$th of the Pell sequences $Q_2(P_G; y, x)$ and $Q_2(P_G; x, y)$ are as follows, respectively:

$$x_{t.hP_2(p)} = y,\ x_{t.hP_2(p)+1} = xy^d$$

and

$$x_{t.hP_2(p)} = xy^d,\ x_{t.hP_2(p)+1} = y$$

where $d$ is an integer such that $tc = d \pmod{2^p - 1}$.

**ii.** The elements $(\alpha)$th of the generalized order-$k$ Pell sequences $Q_k(P_G; y, x)$ and $Q_k(P_G; x, y)$ for all $t.hP_k(p)-(k-2) \le \alpha \le t.hP_k(p)+1$ are in the following forms, respectively:

$$x_{t.hP_k(p)-(k-2)} = y^{c_1},\ x_{t.hP_k(p)-(k-2)+1} = y^{c_2},\dots,\ x_{t.hP_k(p)-1} = y^{c_{k-2}},\ x_{t.hP_k(p)} = y^{c_{k-1}},\ x_{t.hP_k(p)+1} = xy^{c_k},$$

where $c_1, c_2,\dots, c_k$ are integers such that $0 \le c_1, c_2,\dots, c_k < 2^p - 1$ and

$$x_{t.hP_k(p)-(k-2)} = y^{d_1},\ x_{t.hP_k(p)-(k-2)+1} = y^{d_2},\dots,\ x_{t.hP_k(p)-1} = y^{d_{k-2}},\ x_{t.hP_k(p)} = xy^{d_{k-1}},\ x_{t.hP_k(p)+1} = y^{d_k},$$

where $d_1, d_2,\dots, d_k$ are integers such that $0 \le d_1, d_2,\dots, d_k < 2^p - 1$.

**Proof. i.** If $k = 2$, consider the recurrence relations defined by the following:

$P_n$ denote the nth member of the Pell sequence

$$P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 2, \text{ where } P_0 = 0, P_1 = 1;$$

$$v_n = \left(2^{P_{n+1}}.v_{n-2} + v_{n-1}\right).\ 2^{P_{n-1}} + v_{n-1} \text{ for } n \geq 2, \text{ where } v_0 = 1, v_1 = 0$$

and

$$u_n = P_{n-1} \text{ for } n \geq 1, \text{ where } u_0 = 1;$$

$$w_n = \left(2^{u_{n-1}}.w_{n-2} + w_{n-1}\right).\ 2^{u_{n-1}} + w_{n-1} \text{ for } n \geq 2, \text{ where } w_0 = 0, w_1 = 1.$$

Then the Pell sequences $Q_2(P_G; y, x)$ and $Q_2(P_G; x, y)$ are in the following forms, respectively:

$x_0 = y,\ x_1 = x,\ x_2 = yx^2 = x^2y^4, \ldots, x_{hP_2(p)} = x^{P_{hP_2(p)}} y^{v_{hP_2(p)}},\ x_{hP_2(p)+1} = x^{P_{hP_2(p)+1}} y^{v_{hP_2(p)+1}}, \ldots,$

$x_{t.hP_2(p)} = x^{P_{t.hP_2(p)}} y^{v_{t.hP_2(p)}},\ x_{t.hP_2(p)+1} = x^{P_{t.hP_2(p)+1}} y^{v_{t.hP_2(p)+1}}, \ldots,$

and

$x_0 = y,\ x_1 = x,\ x_2 = yx^2 =, \ldots, x_{hP_2(p)} = x^{u_{hP_2(p)}} y^{w_{hP_2(p)}},\ x_{hP_2(p)+1} = x^{u_{hP_2(p)+1}} y^{w_{hP_2(p)+1}}, \ldots,$

$x_{t.hP_2(p)} = x^{u_{t.hP_2(p)}} y^{w_{t.hP_2(p)}},\ x_{t.hP_2(p)+1} = x^{u_{t.hP_2(p)+1}} y^{w_{t.hP_2(p)+1}}, \ldots,$

Using Lemma 3.1 the elements $(hP_2(p))$th and $(hP_2(p)+1)$th of the Pell sequences $Q_2(P_G; y, x)$ and $Q_2(P_G; x, y)$ are obtained as follows, respectively:

$$x_{hP_2(p)} = y,\ x_{hP_2(p)+1} = xy^c$$

and

$$x_{hP_2(p)} = xy^c,\ \ x_{h_2(p)+1} = y,$$

where $c$ is an integer such that $v_{hP_2(p)+1} \equiv c(\text{mod } 2^p - 1)$ and $w_{h_2(p)} \equiv c(\text{mod } 2^p - 1)$.

Also, it is easy to see from Lemma 3.1 that the elements $(t.hP_2(p))$th and $(t.hP_2(p)+1)$th of the Pell sequences $Q_2(P_G; y, x)$ and $Q_2(P_G; x, y)$ are as follows, respectively:

$$x_{t.hP_2(p)} = y\ ,\ x_{t.hP_2(p)+1} = xy$$

and

$$x_{t.hP_2(p)} = xy^{t.c},\ \ x_{t.hP_2(p)+1} = y.$$

If $t.c \equiv d(\text{mod } 2^p - 1)$, we get the elements $(t.hP_2(p))$ th and $(t.hP_2(p)+1)$ th of the Pell sequences $Q_2(P_G; y, x)$ and $Q_2(P_G; x, y)$, respectively

$$x_{t.hP_2(p)} = xy^d,\ \ x_{t.hP_2(p)+1} = y$$

and

$$x_{t.hP_2(p)} = y,\ x_{t.hP_2(p)+1} = xy^d.$$

**ii.** If $k \geq 3$, consider the recurrence relations defined by the following:

$$u_n = 2u_{n-1} + u_{n-2} + \ldots + u_{n-k} \text{ for } n \,\square\, k, \text{ where } u_0 = 0,\ u_1 = 1 \text{ and}$$
$$u_j = 2u_{j-1} + u_{j-2} + \ldots + u_0 \text{ for } 2 \leq j \leq k - 1$$

and

$$v_n = 2v_{n-1} + v_{n-2} + \ldots + v_{n-k} \text{ for } n \geq k, \text{ where } v_0 = 1,\ v_1 = 0 \text{ and}$$
$$v_j = 2v_{j-1} + v_{j-2} + \ldots + v_0 \text{ for } 2 \leq j \leq k - 1.$$

Then the generalized order-$k$ Pell sequences $Q_k(P_G; y, x)$ and $Q_k(P_G; x, y)$ are in the following forms, respectively:

$$x_0 = y,\ x_1 = x,\ x_2 = x^2y^4,\ x_3 = x^5y^{52},\ldots,\ x_k = x^{u_k}y^{\mu},\ldots,$$
$$x_{t.hP_k(p)-k+2} = x^{u_{t.hP_k(p)-k+2}}y^{c_1},\ = x_{t.hP_k(p)-k+3} = x^{u_{t.hP_k(p)-k+3}}y^{c_2},\ldots,$$
$$x_{t.hP_k(p)} = x^{u_{t.hP_k(p)}}y^{c_{k-1}},\ x_{t.hP_k(p)+1} = x^{u_{t.hP_k(p)+1}}y^{c_k},\ldots,$$

where $\mu, c_1, c_2,\ldots, c_k$ are integers such that $0 \le \mu, c_1, c_2,\ldots, c_k < 2^p - 1$ and

$$x_0 = x,\ x_1 = y,\ x_2 = xy^2,\ x_3 = x^3y^{10},\ldots,\ x_k = x^{v_k}y^{\eta},\ldots,$$
$$x_{t.hP_k(p)-k+2} = x^{v_{t.hP_k(p)-k+2}}y^{d_1},\ x_{t.hP_k(p)-k+3} = x^{v_{t.hP_k(p)-k+3}}y^{d_2},\ldots,$$
$$x_{t.hP_k(p)} = x^{v_{t.hP_k(p)}}y^{d_{k-1}},\ x_{t.hP_k(p)+1} = x^{v_{t.hP_k(p)+1}}y^{d_k},\ldots,$$

where $\eta, d_1, d_2,\ldots, d_k$ are integers such that $0 \le \eta, d_1, d_2,\ldots, d_k < 2^p - 1$.

Using Lemma 3.1 the elements $(\alpha)$ th of generalized order-$k$ Pell sequences $Q_k(P_G; y, x)$ and $Q_k(P_G; x, y)$ for all $t.hP_k(p) - (k-2) \le \alpha \le t.hP_k(p) + 1$ are obtained as follows, respectively:

$$x_{t.hP_k(p)-k+2} = y^{c_1}, x_{t.hP_k(p)-k+3} = y^{c_2},\ldots,\ x_{t.hP_k(p)} = y^{c_{k-1}},\ x_{t.hP_k(p)+1} = xy^{c_k},\ldots,$$

and

$$x_{t.hP_k(p)-k+2} = y^{d_1},\ x_{t.hP_k(p)-k+3} = y^{d_2},\ldots,\ x_{t.hP_k(p)} = xy^{d_{k-1}},\ x_{t.hP_k(p)+1} = y^{d_k},\ldots.$$

Lemma 3.2 gives immediately

**Theorem 2.1.** Let $p$ be a prime such that $p \ge 3$.

**i.** If the elements $(hp_2(p))$th and $(hp_2(p)+1)$th of the Pell sequence $Q_2(P_G; y, x)$ are $x_{hP_2(p)} = y$ and $x_{hP_2(p)+1} = xy^c$, respectively, then $PerQ_2(P_G; y, x) = PerQ_2(P_G; x, y) = \sigma.h_2(p)$, where $\sigma$ is the last natural number such that $\sigma.c \equiv 0 \pmod{2^p - 1}$.

**ii.** The generalized order-$k$ Pell sequences $Q_k(P_G; y, x)$ and $Q_k(P_G; x, y)$ are form layers of length $hP_k(p)$.

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