Potentials for the Klein-Gordon and Dirac Equations

Lalit K. Sharma*, Pearson V. Luhanga and Samuel Chimidza
Department of Physics, University of Botswana, Gaborone, Botswana.
*Author for correspondence; e-mail: sharmalk@mopipi.ub.bw

Received: 21 December 2010
Accepted: 19 January 2011

ABSTRACT

A few solvable potentials for the Klein-Gordon equation have been evaluated by using various standard second order differential equations. Some solvable potentials for the Dirac equation have also been discussed.

Keywords: solvable potentials; general methods; klein-gordon equation; dirac equation.

1. INTRODUCTION

In view of the difficulties encountered in quantum field theory, potential theory has always been regarded as a solvable mathematical model giving the general physical behaviour. Thus a study of a wide class of solvable potentials is expected to shed more light to the deeper understanding of these problems. There are only a few kinds of potentials for which the Schrödinger equation can be solved exactly. Of course, a numerical solution is possible for any type of potential.

Kermode [1] outlined a simple numerical method for the s-wave and in a subsequent paper Kermode et al [2] also outlined a model for higher angular momentum states. Sirohi and Shrivastava [3] improved this method and reduced the computational labour involved. Based on the above methods, Iyer and Sharma [4] obtained eigen energy expression for a local potential. However, it should be emphasized here that for the investigation of properties of the scattering amplitude it is always useful to have explicitly solvable examples.

Bargmann [5] initiated the problem of constructing families of solvable potentials for the Schrödinger equation. This author evaluated potentials which gave equal phase shifts for the s-waves, the so called phase equivalent potentials. Bhattacharjie and Sudershan [6], formulated a general method for constructing solvable velocity independent potentials for the Schrödinger equation. Considering the normal form of a second order differential equation and applying the Schwarzian derivative, a few potentials have been constructed by Bose [7]. In each of these papers [6-7], authors investigated the potentials constructed through the hypergeometric equation. Knowing the general solutions of the basic differential equation, one may then study the nature of solutions for the potential appearing in the radial Schrödinger equation. Aly and Spector [8], utilizing the modified Mathieu differential equation and Darboux’s theorem for second-order differential equation, constructed some solvable potentials for the Schrödinger equation. The limitations of the potentials constructed by the Darboux theorem were also discussed by them.
Stillinger [9] has, after making appropriate substitutions, obtained potentials by converting the Schrödinger equation into general differential equations for the parabolic cylinder functions. Further work on the construction of both singular and non-singular potentials for the Schrödinger equation has been carried out by a number of workers [10-14]. It may be noted here that Sharma [11-12] and Sharma et al [13-14], using both the techniques in [6-7] for constructing potentials, have shown that they generate identical potentials, when same transformations are applied in each of the methods.

Sometimes the finite series limit in the polynomials may even inspire the exact and complete analytic solution of the Schrödinger eigenvalue problem [15]. It has also been shown [16] that a broad class of regular potentials may admit exact bound-state solutions in the generalized harmonic oscillator wave function form under certain necessary and sufficient conditions. Ecker and Weizel [17] solved the radial Schrödinger equation by a simple and approximate method. Later Sharma and Chaturvedi [18] using this approximation, obtained bound s-state energies for the superposed screened Coulomb potential. The binding energies have also been evaluated for different atoms for the screened Coulomb potential, using non-perturbative solutions [19-20].

The exact solutions of the non-relativistic and the relativistic equations with the Coulomb potential play a very important role in quantum mechanics. For example the exact solutions of the Schrödinger equation for hydrogen and for the harmonic oscillator in three dimensions [21] are an important milestone in the beginning stage of quantum mechanics, which provided a strong evidence for supporting the correctness of the quantum theory. It should be noted that during the past several decades more attention has been paid on the study of bound-states than on the scattering states, for a given quantum system. Nevertheless, it is always essential to study both of them in order to understand a given quantum system completely. Since the establishment of quantum mechanics, the study of the bound and scattering states of quantum system has been known and understood well in the framework of the Schrödinger equation [22]. For instance, the study of the scattering particles by the Coulomb field has been the subject of interest both in quantum mechanics and in classical mechanics. In the case of the non-relativistic Schrödinger equation, the scattering phase shifts and the differential cross-section have been known for a long time [21].

It may be pointed out here that in relativistic quantum mechanics, the exact solutions of the wave equations are very important for the understanding of physics that can be understood by such solutions. For example, they prove to be valuable tools in determining the radiative contributions to the energy. The relativistic quantum mechanical wave equations are therefore needed essentially for the study of fast electrons scattered by a nucleus in which the relativistic effects have to be considered. Recently, the Dirac equation with position-dependent mass in a Coulomb field [23] has been studied. The exact solutions of the Dirac equation with the Coulomb plus scalar potential in two-dimensions [24-25] and higher dimensions [26] have also been carried out.

Exact solutions of the Dirac equation for two electromagnetic potentials viz., the vector and scalar potentials have been obtained by Sharma and Fiase [27]. They derived bound-state solutions for the Kratzer potential in the framework of Dirac equation. Stanciu [28] has solved the Dirac equation for different configurations of the external magnetic fields. These previously unknown solutions of the
Dirac equation are written in terms of the known solutions of the Schrödinger equation. Sharma et al [29] have also solved Dirac equation for a linear potential, to derive expressions for the wave function and phase shifts.

In this paper, using various standard second order differential equations, a few solvable potentials for the Klein-Gordon and Dirac equations have been constructed. In section 2, general methods have been discussed. In sections 3 and 4, we deal with the construction of potentials for the Klein-Gordon and Dirac equations respectively. A large number of potentials have been constructed. Details of all the potentials derived in this paper are listed in Table 1.

It may be noted that in section 2, we have derived two potentials for the Dirac equation. A careful literature survey reveals that since its formulation, a very few solvable configurations exist for the Dirac equation. Our method, being quite simple and straightforward, can therefore be used in solving the Dirac equation for a few other additional configurations also.

2. General Method

(A) The general form of second order differential equation is given by

\[ W''(z) + p(z)W'(z) + q(z)W(z) = 0 \]  
\[ (1) \]

Setting

\[ z = f(r), W(z) = g(r)\phi(r) \text{ and } g(r) \neq 0. \]  
\[ (2) \]

Equation (1) can be put in the form

\[ \phi''(r) + A(r)\phi'(r) + B(r)\phi(r) = 0 \]  
\[ (3) \]

with

\[ A(r) = \frac{d}{dr} \left[ \ln \frac{\{g(r)\}^2}{f'(r)} \right] p[f'(r)]f''(r) \]  
\[ (4) \]

and

\[ B(r) = \frac{d}{dr} \left[ \frac{g'(r)}{g(r)} \right] - \frac{\{g'(r)\}^2}{g(r)} A(r) + q[f'(r)]\frac{f''(r)}{f'(r)} \]  
\[ (5) \]

Since the Klein-Gordon equation and the Dirac equation in the non-relativistic limit may be reduced to look like Schrödinger equation, the potentials may be constructed by taking \( A(r) = 0 \) and equating the remaining part of the Klein-Gordon equation or the Dirac equation with \( B(r) \). For (3) to be of the form of the radial Schrödinger equation with the scalar potential \( V_s(r) \),

\[ \psi''(r) + \left[ K^2 - V_s(r) - \frac{l(l+1)}{r^2} \right] \psi(r) = 0, \]  
\[ (6) \]

the necessary and sufficient conditions are

\[ A(r) = 0, B(r) = K^2 - V_s(r) - \frac{l(l+1)}{r^2} \frac{\partial}{\partial k} V_s(r) = 0. \]  
\[ (7) \]

Thus, under suitable transformations a linear second order differential equation can be transformed to the Schrödinger equation (6) with the scalar potential \( V_s(r) \).
Table 1. Solvable Potentials for Klein-Gordon Equation.

<table>
<thead>
<tr>
<th>Equation</th>
<th>( f(r) )</th>
<th>( g(r) )</th>
<th>( V(r) )</th>
<th>Scaling factor(h)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Laguerre equation</td>
<td>( W(z) = e^{z/2} z^{-1/2} \cdot v(z) ) &lt;br&gt;( z = f(r) = \alpha r ) &lt;br&gt;( \alpha = \pm iK )</td>
<td>( D \sqrt{\alpha} )</td>
<td>( V(x) = \frac{1}{2x} )</td>
<td>( h = \frac{-F}{2iK(n + \frac{1}{2})} )</td>
<td>( l = 0 )</td>
</tr>
<tr>
<td>( zW''(z) + (1 - z)W'(z) + ) &lt;br&gt;( nW(z) = 0 )</td>
<td>( f(r) = \tan^{-1} \left( \frac{\cos \alpha r}{1 + \sin \alpha r} \right) ) &lt;br&gt;( \alpha = \pm 2iK )</td>
<td>( D \sqrt{\alpha} )</td>
<td>( V(x) = \frac{\alpha h}{2 \tan^{-1} \left( \frac{\cos \alpha x}{1 + \sin \alpha x} \right)} )</td>
<td>( h = \frac{-E}{2iK(n + \frac{1}{2})} )</td>
<td>( l = 0 )</td>
</tr>
<tr>
<td>2. Associated Legendre equation</td>
<td>( 1 - z^2 ) ( \frac{W''(z) - 2zW'(z) + \frac{m^2}{n(n + 1)} W(z)}{n(n + 1) - \frac{m^2}{(1 - z^2)}} \cdot W(z) = 0 )</td>
<td>( \rho = \frac{1}{2} \ln \frac{1 + z}{1 - z} )</td>
<td>( V(r) = \frac{\alpha \left( 1 - m^2 r^2 \right)}{1 - \alpha^2 r^2} )</td>
<td></td>
<td>( l = 0 )</td>
</tr>
<tr>
<td>( K = 0 )</td>
<td>( \rho = f(r) = \frac{1}{2} \ln \frac{1 + \alpha r}{1 - \alpha r} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Tschebyscheff equation</td>
<td>( \frac{1 - z^2}{n^2} ) ( W''(z) - zW'(z) + ) &lt;br&gt;( n^2 W(z) = 0 )</td>
<td>( z = f(r) = \alpha r ) &lt;br&gt;( K = 0 )</td>
<td>( D \left( \frac{\alpha}{1 - \alpha^2 r^2} \right)^{1/2} )</td>
<td>( V(r) = \frac{\alpha}{1 - \alpha^2 r^2} )</td>
<td>( l = 0 )</td>
</tr>
<tr>
<td>( 4. Stokes equation )</td>
<td>( W''(z) + bzW(z) = 0 )</td>
<td>( z = f(r) = \frac{(\alpha r)^{1/3}}{\sqrt[3]{3}} ) &lt;br&gt;( K = 0 )</td>
<td>( D \frac{\alpha^{1/6} r^{-1/3}}{\sqrt[3]{3}} )</td>
<td>( V(r) = \frac{\sqrt[3]{2}}{3r} )</td>
<td>( l = 0 )</td>
</tr>
<tr>
<td>( z = f(r) = \left( \frac{\alpha}{r} \right)^{2/3} ) &lt;br&gt;( K = 0 )</td>
<td>( D \left( \frac{2}{3} \alpha^{1/3} r^{-5/6} \right) )</td>
<td></td>
<td>( V(r) = \frac{2 \alpha \sqrt{b}}{3r^2} )</td>
<td></td>
<td>( l = 0 )</td>
</tr>
<tr>
<td>Equation</td>
<td>f(r)</td>
<td>g(r)</td>
<td>V(r)</td>
<td>Scaling factor(h)</td>
<td>Remarks</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------------------</td>
<td>---------</td>
</tr>
<tr>
<td>5. Whittaker equation</td>
<td>( z = f(r) = \alpha r )</td>
<td></td>
<td></td>
<td>( h = \frac{\sqrt{x}}{iKK'} )</td>
<td>( l = 0 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = \pm 2iK )</td>
<td>( D\sqrt{\alpha} )</td>
<td>( V(x) = \left( \frac{1}{x} \right)^{\frac{1}{2}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( W''(z) + \left[ -\frac{1}{4} + \frac{k'}{z} + \frac{\alpha^2 - m^2}{z^2} \right] W(z) = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Bessel equation</td>
<td>( f(r) = \alpha e^{-\varphi r} )</td>
<td>( \alpha = \pm \frac{K}{2m} )</td>
<td>( V(x) = ahe^{-2ax} )</td>
<td>( h = \frac{4imE}{KK'} )</td>
<td>( l = 0 )</td>
</tr>
<tr>
<td></td>
<td>( D\sqrt{2i\alpha e^{-\varphi r}} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( W''(z) + \left( 1/z \right)W'(z) + \left( 1 - m^2/z^2 \right)W(z) = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Sharpe equation</td>
<td>( f(r) = \alpha r^{1/2} )</td>
<td>( K = 0 )</td>
<td>( V(x) = \frac{1}{2x} )</td>
<td></td>
<td>( J(l+1) = \frac{\alpha^2}{4} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{D}{\sqrt{2r}} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( W''(z) + \left( 1/z \right)W'(z) + \left( z + A \right)W(z) = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Hermite equation</td>
<td>( f(r) = \alpha )</td>
<td>( \alpha = \pm \frac{K}{2} )</td>
<td>( V(x) = \frac{\sqrt{3}}{4x} )</td>
<td>( h = \frac{\sqrt{3}E}{(2n+1)K} )</td>
<td>( l = 0 )</td>
</tr>
<tr>
<td></td>
<td>( D\sqrt{\frac{1}{2} \alpha^2} )</td>
<td>( \alpha = \pm 2K )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( W''(z) - 2zW'(z) + 2nW(z) = 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(B) Bose Method

In this method the standard second order differential equation is first reduced to normal form as

\[ v''(Z) + l(Z)v(Z) = 0 \]  \hspace{1cm} (8)

Now setting

\[ Z = Z(r) \]  \hspace{1cm} (9)

and

\[ v(z) = (z')^2 \phi(r) \]  \hspace{1cm} (10)

where \( Z' \) is the first derivative of \( Z(r) \) with respect to \( r \), transforms equation (8) to the form

\[ \phi''(r) + \left[ z'^2 l(z) + \frac{1}{2} \{z,r\} \right] \phi(r) = 0 \]  \hspace{1cm} (11)

where \( \{z,r\} \) is the Schwarzian derivative defined in [30]

\[ \{z,r\} = \frac{z''(r)}{z'(r)} - \frac{3}{2} \left( \frac{z'(r)}{z''(r)} \right)^2 \]  \hspace{1cm} (12)

Comparison of equation (11) in the non-relativistic limit of the Klein-Gordon equation yields

\[ (Z')^2 l(Z) + \frac{1}{2} \{Z,r\} = I_s(r) = K^2 - 2EV + V^2 - \frac{l(l+1)}{r^2} \]  \hspace{1cm} (13)

where for the Schrödinger equation

\[ I_s(r) = K^2 - V_s(r) - \frac{l(l+1)}{r^2} \]  \hspace{1cm} (14)

Equation (13) thus enables us to give the form of the potential to be constructed. It may be noted that in both the methods [A] and [B], the units chosen are \( \hbar = 2 \hbar = 1 \) such that the nonrelativistic energy \( E' = K^2 \).

2. Potentials from the Klein-Gordon equation

We now consider the Klein Gordon wave equation [31] of the form

\[ \left[ i \frac{\partial}{\partial t} - e\phi \right]^2 - \left( \frac{1}{i} \nabla - eA \right)^2 \Psi = m^2 \Psi. \]  \hspace{1cm} (15)

Equation (15) represents a spinless particle of charge \( e \) and mass \( m \) in a scalar potential \( \phi \) and vector potential \( A(r,t) \), where natural units (\( \hbar = c = 1 \)) have been used. If we consider the vector potential \( A(r,t) = 0 \) and assume the scalar potential to be time independent, equation (15) is reduced to the form

\[ \left[ \nabla^2 - \frac{\partial^2}{\partial t^2} - 2ie\phi \frac{\partial}{\partial t} + e^2 \phi^2 \right] \Psi = m^2 \Psi \]  \hspace{1cm} (16)

Setting

\[ \Psi(r,t) = \psi(r)e^{-\frac{it}{\hbar}} \]

equation (16) takes the form

\[ \nabla^2 (E - e\phi)^2 \psi(r) = m^2 \psi(r) \]  \hspace{1cm} (17)

This is a well-known form of the Klein-Gordon equation [32].
Further separating the variables as
\[ \psi(r, \theta, \phi) = R(r) Y_m(\theta, \phi), \]  
equation (17) finally gets transformed to the following radial form with the scalar potential \( V = e^\phi \) for the Klein-Gordon equation
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left( E^2 - m^2 \right) + \left( V - 2E \frac{r}{r^2} \right) R = 0, \]
\[ l = 0, 1, 2, 3, \ldots \]  
(19)

Substituting \( R = \frac{\psi(r)}{r} \) in equation (19), one gets
\[ \psi''(r) + \left[ K^2 - 2Ev + \frac{l(l+1)}{r^2} \right] \psi(r) = 0, \]
(20)
where \( E^2 = K^2 + m^2 \).

Equation (20) is equivalent to Schrödinger equation (6) with \( V_s \) replaced by \( (2Ev - V^2) \).

Now to construct the potentials, general methods (A) and (B) are applied to various second order differential equations.

(A) Stokes equation

The Stokes equation [33] has the form
\[ W''(z) + bzW(z) = 0 \]  
(21)
For this equation, condition \( A(r) = 0 \) yields
\[ g(r) = D \left( \frac{f'(r)}{r} \right)^2 \]  
(22)
where \( D \) is a constant.

Now with the choice of
\[ z = f(r) = \left( \frac{\alpha}{r} \right)^2, \quad g(r) = iD \sqrt{3} \frac{1}{\alpha} \frac{5}{r^6}, \quad K = 0, \quad l = 0 \]  
(23)
and equating the coefficients of \( \psi(r) \) in equation (20) with \( B(r) \), the following inverse square potential of the form
\[ V(r) = \frac{2\alpha \sqrt{b}}{3r^2}, \]  
(24)
is obtained.

Vasudevan et al [10] have obtained a similar potential from Mathieu equation with a suitable choice. It may also be noted that the solution of the Klein-Gordon equation for an inverse square potential is similar to the solution of the Schrödinger equation for an inverse fourth power potential.

Further it is found that the same potential can also be constructed by the Bose technique.

For the Stokes equation
\[ I(z) = bz \]  
(25)
so that for the choice of equation (23)
\[ I_s(r) = \frac{4b\alpha^2}{9r^5} + \frac{5}{36r^2} \]  
(26)
Identifying this with $B(r)$, potential (24) is generated for the energy $E$ given as

$$E = -\frac{5}{48\alpha \sqrt{b}}$$  \hspace{1cm} (27)

Thus it shows that two techniques (A) and (B) are equivalent as they give similar potentials for the same choice of $z = f(r)$ for the Klein-Gordon equation. It may be interesting to note that one of the present authors, while working on solvable potential problems on the Schrödinger equation [11-12], has shown that the two methods construct exactly the same type of potentials, for the same choice of $z = f(r)$ for this case as well.

(B) Laguerre equation

The Laguerre equation [34] is

$$ZW''(z) + (1 - Z)W'(z) + nW(z) = 0$$  \hspace{1cm} (28)

Setting

$$W(z) = e^z z^{-1/2} v(z),$$  \hspace{1cm} (29)

equation (28) reduces to the form [14]

$$v''(z) + \left( \frac{1}{4z^2} + \frac{n + 1/2}{z} - \frac{1}{4} \right) v(z) = 0$$  \hspace{1cm} (30)

Applying the general method with the choice

$$z = f(r) = \tan^{-1}\left( \frac{\cos \alpha r}{1 + \sin \alpha r} \right), \quad g(r) = iD\sqrt{\alpha} \quad \text{and} \quad \alpha = \pm 2iK.$$  \hspace{1cm} (31)

the following potential is obtained for the s-wave Klein-Gordon equation

$$V(r) = \frac{\alpha}{2 \tan^{-1}\left( \frac{\cos \alpha r}{1 + \sin \alpha r} \right)}$$  \hspace{1cm} (32)

This potential is energy dependent since, $\alpha = \pm 2iK$. To make it energy-independent, the use of the following scale transformation is made

$$r = h x,$$  \hspace{1cm} (33)

where $h$ is a scaling factor. With the above scale transformation the potential has the form

$$V(x) = \frac{ah}{2 \tan^{-1}\left( \frac{\cos \alpha h x}{1 + \sin \alpha h x} \right)}$$  \hspace{1cm} (34)

where the scaling factor is

$$h = \frac{-E}{2iK \left( n + \frac{1}{2} \right)}$$  \hspace{1cm} (35)
(C) Whittaker equation

This equation has the form [35]

\[ W^*(z) + \left[ -\frac{1}{4} + \frac{K'}{z} + \frac{1}{4} \frac{m^2}{z^2} \right] W(z) = 0. \tag{36} \]

It is found that the exponential type of potential is generated from this equation with the choice

\[ z = f(r) = i e^{-\alpha r}, \quad g(r) = D \sqrt{-2i\alpha} e^{-\alpha r}, \quad \alpha = \pm \frac{K}{2m} \text{ and } l = 0. \tag{37} \]

Here the exponential potential thus obtained is also energy dependent. Hence after scaling, the potential is of the form

\[ V(x) = \alpha h e^{-2\alpha h x} \tag{37} \]

with the following choice of scaling factor

\[ h = \frac{4i \alpha \hbar}{K K'} \tag{38} \]

(D) Associated Legendre equation

This equation [36] can be written as

\[ (1-z^2)W^*(z) - 2z W'(z) + \left[ n(n+1) - \frac{m^2}{(1-z^2)} \right] W(z) = 0. \tag{39} \]

Using the transformation

\[ \rho = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right), \tag{40} \]

equation (39) is transformed to

\[ W^*(\rho) + \left[ \frac{4n(n+1)e^{2\rho}}{(e^{2\rho} + 1)^2} - m^2 \right] W(\rho) = 0. \tag{41} \]

For the choice

\[ \rho = f(r) = \frac{1}{2} \ln \left( \frac{1+\alpha r}{1-\alpha r} \right), \]
\[ g(r) = \frac{c\sqrt{\alpha}}{(1-\alpha^2r^2)^{1/2}} \quad \text{and} \quad K = 0 \tag{42} \]
leads to the potential (for s-waves)

\[ V(r) = \frac{(1-m^2)^{1/2}\alpha}{(1-\alpha^2 r^2)} \]  \hspace{1cm} (43)

and

\[ E = \frac{-n(n+1)\alpha}{2(1-m^2)^{1/2}} \]  \hspace{1cm} (44)

It may be interesting to note here that the same type of potential (43) may also be generated from Tschebyscheff polynomial [35] given as

\[ W''(z) - zW'(z) + n^2W(z) = 0, \]  \hspace{1cm} (45)

with the choice

\[ z = f(r) = \alpha r, \]

\[ g(r) = D\left\{\frac{\alpha}{(1-\alpha^2 r^2)^{1/2}}\right\}^{1/2} \]  \hspace{1cm} (46)

The solutions of the Tschebyscheff polynomial are given in terms of confluent hypergeometric functions. Vasudevan et al [10] have shown that potential (43) can also be generated from spheroidal equation with suitable transformations.

Finally, particular mention may be made that for the s-waves the Coulomb potential can be constructed from various differential equations like Sharpe, Hermite, Bessel, Laguerre and Whittaker equations with a suitable choice of \( f(r) \). For the Klein-Gordon equation, Vasudevan et al [10] have also obtained potential (43) from Weber-Hermite differential equation. However, one must note that it is for the Bessel equation only that the Coulomb potential can be generated for the values of \( l \neq 0 \). Also in this case the potential obtained is energy independent.

Various potentials constructed in this paper for the Klein-Gordon equation are listed in Table I. Suitable choices of \( f(r) \) and \( g(r) \), used in the derivation of these potentials have also been shown in Table 1.

2. Potentials for the Dirac equation

The Dirac equation in the non relativistic limit can be written as [37]

\[ E'\psi_\alpha (r) = \left[ -\nabla^2 + V(r) - \frac{dV}{dr} \frac{\partial}{\partial r} - \nabla^4 + \frac{2}{r} \frac{dV}{dr} \hat{S} \cdot \hat{L} \right] \psi_\alpha (r). \]  \hspace{1cm} (47)

Equation (47) with only the first two terms on the right hand side looks like the Schrödinger equation. The last term represents the spin-orbit coupling energy, while \( \psi_\alpha \) is a two-component spinor and

\[ E' = E - m \]  \hspace{1cm} (48)

Equation (47) can be solved for the Schrödinger part with the first two terms on the right hand side. The usual method is to solve the Schrödinger part of the equation and consider the last three terms as perturbation. Here, however, we consider some potentials for which closed form solutions exist even with the addition of the third term \( -\frac{dV}{dr} \frac{\partial\psi_\alpha}{\partial r} \) to the Schrödinger
part. The last two terms could then be considered as perturbations. We then have to solve (for small $K$), the following equation

$$
\Psi'_A(r) + V'(r)\Psi'_A(r) + \left( K^2 - V - \frac{l(l+1)}{r^2} \right)\Psi_A(r) = 0
$$

(49)

With the substitutions

$$
\Psi_A = \exp\left( -\frac{V(x)}{2} \right)\Phi(x)
$$

(50)

and

$$
r = \sqrt{D} \cdot x
$$

(51)

where $D$ is a constant, equation (49) changes over to

$$
\Phi'(x) + \left[ D(k^2 - V(x)) - \frac{V''(x)}{4} - \frac{V(x)}{2} - \frac{l(l+1)}{x^2} \right] \Phi(x) = 0
$$

(52)

The above is the general form of the equation from which potentials can be constructed, utilizing various standard differential equations.

For example a potential, $V(x) = V_0 \left( 1 + \frac{a}{V_0} \right) x^2$, can be obtained from equation (52) when it is transformed to the Whittaker equation (36), by setting

$$
z = \left( b^2 x^2 \right)^{1/4}, \phi(x) = z^{-3/4} W(z)
$$

(53)

and with the choice

$$
Dk^2 = V_0, a^7 b^4 = 1, K' = \frac{3ab}{4}
$$

(54)

and

$$
1 - m^2 = \frac{3}{16} - \frac{Da}{4}.
$$

Similarly a potential of exponential form may also be generated from Laguerre equation with a suitable choice for $f(r)$.

Finally it will be worth mentioning here that the present authors, using various standard differential equations, have recently constructed solvable potentials for the Dirac equation in the non-relativistic limit. This work has since been accepted for publication in the Journal of Mathematical Sciences [38].
REFERENCES


[34] Sneddon I.N. Special functions of mathematical physics and chemistry, Oliver & Boyd.1961.