Infinitesimal Automorphisms in the Tangent Bundle of a Riemannian Manifold with Horizontal Lift of Affine Connection

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ABSTRACT
The main purpose of the present paper is to study conditions for a vertical infinitesimal affine transformation in the tangent bundle of a Riemannian manifold with respect to the horizontal lift of affine connection and then to apply the results obtained to the study of fibre-preserving infinitesimal affine transformation and also to investigate infinitesimal isometry in this setting.

Keywords: lift, tangent bundle, infinitesimal affine transformation, fibre-preserving transformation, infinitesimal isometry.

1. INTRODUCTION
Let $M^n$ be a Riemannian manifold with metric $g$ whose components in a coordinate neighborhood $U$ are $g_{ij}$ and denote by $\Gamma^h_{ij}$ the Christoffel symbols formed with $g_{ij}$. If, in the neighborhood $\pi^{-1}(U)$ of the tangent bundle $T(M^n)$ over $M^n$, $U$ being a neighborhood of $M^n$, then $^h g$ has components given by

$$^h g = \begin{pmatrix} \Gamma^i_{ij} g_{ij} + \Gamma^x_{ij} g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

with respect to $(x^i, y^j)$ induced coordinates in $T(M^n)$ and $\Gamma^h_{ij} = y^j \Gamma^h_{ij}$, $\Gamma^h_{ij}$ being components of the affine connection in $M^n$.

Let be a pseudo-Riemannian metric, then the horizontal lift $^h g$ of $g$ with respect to $\nabla$ is a pseudo-Riemannian metric in $T(M)$. Since $^h g$ is defined by $^h g = ^c g - \gamma(\nabla g)$, where $\gamma(\nabla g)$ is a tensor field of type $, \gamma(\nabla g) = \begin{pmatrix} y^i \nabla_{x^j} g_{ij} & 0 \\ 0 & 0 \end{pmatrix}$, we have $^h g$ and $^c g$ coincide if and only if $\nabla g = 0 [1, p.105]$.

If we write $ds^2 = g_{ij} dx^i dx^j$ the pseudo-Riemannian metric in $M^n$, given by $g$, then the pseudo-Riemannian metric in $T(M)$ given by the $^h g$ of $g$ to $T(M)$ with respect to an affine connection $\nabla$ in $M^n$ is

$$ds^2 = 2 g_{ij} \tilde{y}^i dx^j,$$

where $\tilde{y}^i = dy^i + \tilde{\Gamma}^i_{ij} y^j dx^k$ and $\tilde{\Gamma}^i_{ij} = \Gamma^h_{ij}$ are components of the connection $\tilde{\nabla}$ defined by
We shall now define the horizontal lift $^H\nabla$ of affine connection $\nabla$ in $M_n$ to $T(M_n)$ by the conditions

\begin{align}
^h\nabla^X_Y &= 0, \\
^h\nabla^n_Y &= 0,
\end{align}

(2)

for $X, Y \in \mathfrak{X}_0(M_n)$. From (2), the horizontal lift $^H\nabla$ of $\nabla$ has components $^H\Gamma^k_{ij}$ such that

\begin{align}
^H\Gamma^k_{ij} &= \Gamma^k_{ij}, \\
^H\Gamma^k_{i\beta} &= \Gamma^k_{i\beta}, \\
^H\Gamma^k_{ij} &= \Gamma^k_{ij}, \\
^H\Gamma^k_{ij} &= \Gamma^k_{ij}, \\
^H\Gamma^k_{ij} &= \Gamma^k_{ij},
\end{align}

(3)

with respect to the induced coordinates in $T(M_n)$, where $\Gamma^k_{ij}$ are components of $\nabla$ in $M_n$.

Let $g$ and $\nabla$ be, respectively, a pseudo-Riemannian metric and an affine connection such that $^g\nabla = 0$. Then $^H\nabla^g = 0$, where $^H\nabla^g$ is a pseudo-Riemannian metric. The connection $^H\nabla$ has nontrivial torsion even for the Riemannian connection $\nabla$ determined by $g$, unless $g$ is locally flat [1, p.111].

Let there be given an affine connection $\nabla$ and a vector field $X \in \mathfrak{X}_0(M_n)$. Then the Lie derivative $L_X\nabla$ with respect to $X$ is, by definition, an element of $\mathfrak{X}_0(M_n)$ such that

\begin{align}
(L_X\nabla)(Y, Z) &= L_X(\nabla_Y Z) - \nabla_Y (L_X Z) - \nabla_{[X,Y]} Z = [L_X, \nabla_Y] Z - \nabla_{[X,Y]} Z,
\end{align}

(4)

for any $Y, Z \in \mathfrak{X}_0(M_n)$.

In a manifold $M_n$ with an affine connection $\nabla$, an infinitesimal affine transformation $x^k = x^k + X^k(x^1, \ldots, x^n)\Delta t$ defined by a vector field $X \in \mathfrak{X}_0(M_n)$ is called an infinitesimal affine transformation if $L_X\nabla = 0$ [1, p.67].

The main purpose of the present paper is to study the infinitesimal affine transformation and infinitesimal isometry in $T(M_n)$ with affine connection $^H\nabla$.

2. Vertical infinitesimal affine transformations in a tangent bundle with $^H\nabla$

From (4) we see that, in terms of components $\Gamma^\alpha_{\beta\gamma}$ of $\nabla$, $X$ is an infinitesimal affine transformation in $n$-dimensional manifold $M_n$ if and only if,

\begin{align}
\partial_\tau \partial_\beta X^\alpha + X^\lambda \partial_\lambda \Gamma^\alpha_{\beta\gamma} - \Gamma^\lambda_{\gamma\beta} \partial_\lambda X^\alpha + \Gamma^\alpha_{\lambda\gamma} \partial_\lambda X^\lambda + \Gamma^\alpha_{\gamma\lambda} \partial_\beta X^\lambda = 0, \quad \alpha, \beta, \ldots = 1, \ldots, n.
\end{align}

(5)

Let there be given in $M_n$ with an affine connection $\nabla$ with Christoffel symbols $\Gamma^k_{ij}$.
Let \( \hat{X} = \hat{X}^i \partial_i + \hat{X}^k \partial_k \), where \( \partial_i = \frac{\partial}{\partial x^i} \), \( \partial_k = \frac{\partial}{\partial x^k} \), \( T = n + 1, \ldots, 2n \) be a vector field in \( T(M_n) \). Then, taking account of (3), we can easily see from (5) that \( \hat{X} \) is an infinitesimal affine transformations in \( T(M_n) \) with \( ^n \nabla \) if and only if the following conditions (6)-(13) hold:

\[
\begin{align*}
\partial_j \partial_i \hat{X}^h + \hat{X}^k \partial_j \Gamma^h_{ij} + & (\Gamma^h_{ji} \partial_k \hat{X}^i + \partial \Gamma^h_{ji} \partial_k \hat{X}^i) + \Gamma^h_{jk} \partial_i \hat{X}^k + \Gamma^h_{ik} \partial_j \hat{X}^k + y^h R^k_{ij} \partial_i \hat{X}^k = 0, & (6) \\
\partial_j \partial_i \hat{X}^h - & \Gamma^h_{ji} \partial_k \hat{X}^k + \Gamma^h_{jk} \partial_i \hat{X}^k = 0, & (7) \\
\partial_j \partial_i \hat{X}^h - \Gamma^h_{ji} \partial_k \hat{X}^k + & \Gamma^h_{jk} \partial_i \hat{X}^k = 0, & (8) \\
\partial_j \partial_i \hat{X}^h = & 0. & (9)
\end{align*}
\]

\[
\begin{align*}
\partial_j \partial_i \hat{X}^\xi + (\hat{X}^\xi \partial_j \partial_i \hat{X}^h + \hat{X}^\xi \partial_i \partial_j \hat{X}^h) - & (\Gamma^\xi_{ji} \partial_k \hat{X}^i + \partial \Gamma^\xi_{ji} \partial_k \hat{X}^i) + (\partial \Gamma^\xi_{ik} \partial_j \hat{X}^k + \Gamma^\xi_{ik} \partial_j \hat{X}^k) + & \\
+ & (\partial \Gamma^\xi_{jk} \partial_i \hat{X}^k + \Gamma^\xi_{jk} \partial_i \hat{X}^k - \hat{X}^\xi R^h_{ij} - y^\xi \partial \hat{X}^i R^h_{ij} + \partial \hat{X}^i R^h_{ij} \partial_i \hat{X}^k - \hat{X}^\xi R^h_{ij} \partial_i \hat{X}^k - y^\xi \hat{X}^i R^h_{ij} \partial_i \hat{X}^k = 0 & (10)
\end{align*}
\]

\[
\begin{align*}
\partial_j \partial_i \hat{X}^\xi + \hat{X}^\xi \partial_k \partial_j \hat{X}^h - & \Gamma^\xi_{ji} \partial_k \hat{X}^k + \Gamma^\xi_{jk} \partial_i \hat{X}^k + (\partial \Gamma^\xi_{ik} \partial_j \hat{X}^k + \Gamma^\xi_{ik} \partial_j \hat{X}^k - \hat{X}^\xi R^h_{ij} - y^\xi \partial \hat{X}^i R^h_{ij} \partial_i \hat{X}^k = 0, & (11) \\
\partial_j \partial_i \hat{X}^\xi + \hat{X}^\xi \partial_k \partial_j \hat{X}^h - & \Gamma^\xi_{ij} \partial_k \hat{X}^h + \partial \Gamma^\xi_{ik} \partial_j \hat{X}^k + \Gamma^\xi_{ik} \partial_j \hat{X}^k + y^\xi \hat{X}^i R^h_{ij} \partial_i \hat{X}^k = 0, & (12) \\
\partial_j \partial_i \hat{X}^\xi - \Gamma^\xi_{ji} \partial_k \hat{X}^k + & \Gamma^\xi_{jk} \partial_i \hat{X}^k = 0 & (13)
\end{align*}
\]

Let \( \hat{X} \) be a vertical infinitesimal affine transformation in \( T(M_n) \). Then \( \hat{X} \) has components \( \begin{pmatrix} 0 \\ \hat{X}^h \end{pmatrix} \) with respect to the induced coordinates. Thus, from (13), we have

\[
\partial_j \partial_i \hat{X}^h = 0, i.e.,
\]

\[
\hat{X}^h = C^h_i y^i + D^h, \quad (14)
\]

where \( C^h_i \) and \( D^h \) depend only on variables \( \hat{X}^h \). Since \( \hat{X} \) is a vector field in \( T(M_n) \),

\[
C = C_i \partial_i \otimes dx^i \quad \text{and} \quad D = D^h \partial_h
\]

are defined elements of \( \mathfrak{X}_1^1(M_n) \) and \( \mathfrak{X}_0^1(M_n) \), respectively.

**Theorem 1.** If \( \hat{X} \) is a vertical infinitesimal affine transformation of \( T(M_n) \) with \( ^n \nabla \), then

\[
L_D \nabla + C(D \otimes R) = 0, \quad D = \partial^h \frac{\partial}{\partial x^h}, \quad D \in \mathfrak{X}_0^1(M_n) \quad \text{and} \quad C(D \otimes R) = D^h R^h_{bij}.
\]
(b) \( C \) is parallel with respect to \( \nabla \), i.e., \( \nabla C = 0 \)

(c) \( C(T(Y, Z)) = T(CY, Z) = T(Y, CZ) \), for any \( Y, Z \in S^1_\theta(M_n) \), where \( T \) denotes the torsion tensor of \( \nabla \), i.e. \( T \) is pure tensor with respect to \( C \).

(d) \( C(\nabla Z T)(Y, W) = (\nabla_{CZ} T)(Y, W) \), for any \( Y, Z, W \in S^1_\theta(M_n) \).

(e) Conversely, if \( C \) and \( D \) satisfy the conditions (a), (b), (c) and (d), the vector field

\[ \tilde{X} = (C_i^j y^j + D^k) \frac{\partial}{\partial y^k} = \gamma C + \gamma^j D \]

is an infinitesimal affine transformation of \( T(M_n) \) with connection \( \nabla \), where \( \gamma C \) is a vertical vector field, which has components of the form \( \gamma C = \begin{pmatrix} 0 \\ y^i C_i^h \end{pmatrix} \).

**Proof:** (a) Substituting (14) and \( \tilde{X}^h = 0 \) in (10), we have

\[ \partial_j \partial_i C_s^h + C^k_s \partial_k \Gamma^h_{ji} - \Gamma^h_{ji} \partial_k C^h_s - \partial_s \Gamma^h_k - C^k_s + \Gamma^h_{ki} \partial_j C^h_s + \Gamma^h_{kj} \partial_i C^h_s - C^k_s R^h_{yi} + R^h_{yi} C^h_k = 0, \]  

(15)

and

\[ \partial_j \partial_i D^h + D^k \partial_k \Gamma^h_{ji} - \Gamma^h_{ji} \partial_k D^h + \Gamma^h_{ki} \partial_j D^h + \Gamma^h_{kj} \partial_i D^h - D^k R^h_{yi} = 0, \]

(16)

which means that \( L_D \nabla + C(D \otimes R) = 0 \).

(b) Substituting (14) \( \tilde{X}^h = 0 \) and in (12), we obtain,

\[ \partial_j C_i^h - \Gamma^h_{ji} C^h_k + \Gamma^h_{ki} C^h_i = 0, \]

(17)

Substituting (14) and \( \tilde{X}^h = 0 \) in (11), we obtain,

\[ \partial_j C_i^h - \Gamma^h_{ji} C^h_k + \Gamma^h_{ki} C^h_i = 0, \]

(18)

which means \( C \) is parallel in \( M_n \).

(c) Interchanging \( i \) and \( j \) in (18), we have,

\[ \partial_j C_i^h - \Gamma^h_{ij} C^h_k + \Gamma^h_{ik} C^h_i = 0, \]

and subtracting the resulting equation from (17), we have,

\[ T^h_{ji} C^h_k = T^h_{ki} C^h_j, \]

(19)

that is,

\[ C(T(Y, Z)) = T(CY, Z) \]

(20)

for any \( Y, Z \in S^1_\theta(M_n) \). From (19), we obtain \( T(Y, CZ) = C(T(Z, Y)) = C(T(Y, Z)) \) and hence

\[ C(T(Y, Z)) = T(CY, Z) = T(Y, CZ) \]

which is the formula (c).
Using (17) and (18), we eliminate all partial derivatives of $C^h_j$ from (15). Then we obtain,

$$C^h_i \nabla_j T^k_{hi} = \nabla_k T^h_{ij} C^k_j$$, i.e. $T$ is $\phi$ - tensor with respect to $C$.[3].

If we assume that the conditions (a), (b), (c) and (d) are established, then we see that $\tilde{X}$, given in (e), is an infinitesimal affine transformation. Consequently, Theorem 1 is completely proved.

**Theorem 2.** Let $C$ be as in Theorem 1. If $X$ is an infinitesimal affine transformation of $M_n$ with affine connections $\nabla$ and $R(X,Y,Z;\xi)$ is pure with respect to $X$ and $\xi$, so is $CX$.

### 3. Fibre-preserving infinitesimal affine transformation with $^H\nabla$

A transformation of $T(M_n)$ is said to be fibre-preserving if it sends each fibre of $T(M_n)$ into a fibre. An infinitesimal transformation of $T(M_n)$ is said to be fibre-preserving if it generates a local 1-parameter group of fibre-preserving transformations. An infinitesimal transformation $\tilde{X}$ with components $\left(\tilde{x}^h, \tilde{y}^h\right)$ is fibre-preserving if and only if $\tilde{x}^h (h=1,2,\ldots, n)$ depend only on the variables $x^1,\ldots,x^n$ with respect to the induced coordinates $(x^h, y^h)$ in $T(M_n)$. From

$$\begin{cases} x^h = x^h + \tilde{x}^h (x^1,\ldots,x^n) \Delta t \\ x^\tilde{y} = x^\tilde{y} + \tilde{y}^h (x^1,\ldots,x^n, x^1,\ldots,x^n) \Delta t \end{cases}$$

we see that a fibre-preserving infinitesimal transformation $\tilde{X}$ with components $\left(\tilde{x}^h, \tilde{y}^h\right)$ induces an infinitesimal transformation $X$ with components $\tilde{x}^h$ in the base space $M_n$. Since $\partial^h_{j^l} \partial^e_{e^k} \tilde{x}^h = 0$ and $y^j R^k_{j^l} \partial^e_{e^k} \tilde{x}^h = 0$, then from (6) we have:

**Theorem 3.** If $\tilde{X}$ a fibre-preserving infinitesimal transformation of $T(M_n)$ with horizontal lift $^H\nabla$ of a affine connection $\nabla$ in $M_n$ to $T(M_n)$, then the infinitesimal transformation $X$ induced on $M_n$ from $\tilde{X}$ is also affine with respect to $\nabla$.

**Theorem 4.** Let $\nabla$ be a affine connection in $M_n$. Then,

$$(L_X^h \nabla)^c Y, C Z = (L_X^h \nabla)^c Y, C Z + \gamma(L_X R)(, Y, Z),$$ for any $X \in \mathfrak{S}_0^1(M_n)$. 

Proof. Our proposition follows from the following computations:

\[
(L^c_X H \nabla)^c(Y, Z) = L^c_X (H \nabla^c_{c Y} Z) - H \nabla^c_{c Y} (L^c_X Z) - H \nabla^c_{[X, Y]} Z
\]

\[
= L^c_X [\nabla_{c Y} Z - \gamma(R(, Y)Z)] - H \nabla^c_{c Y} (L_X Z) - H \nabla^c_{[X, Y]} Z
\]

\[
= [X, c Y] - [X, \gamma(R(, Y)Z)] - (\nabla_{c Y} (L_X Z)) + \gamma(R(, Y)L_X Z)
\]

\[
= \nabla_{c Y} (L_X Z) + \gamma(R(, Y)Z)
\]

\[
= \nabla_{c Y} (L_X Y) + \gamma(-L_X R(, Y)Z + R(, Y)L_X Z + L_X Y) Z)
\]

\[
= \nabla_{c Y} (L_X Y) + \gamma(L_X R(, Y)Z)
\]

where \( R(, Y) \) denotes a tensor field \( W \) of type (1,1) in \( M \) such that \( W(Z) = R(Z, X)Y \) for any \( Z \in \mathcal{S}_0(M) \). Let \( \tilde{X} \) and \( X \) be as in Theorem 3. From Theorem 4 we see that, it is also known that if \( X \) is infinitesimal affine transformation, \( L_X R = 0 [5] \), then \( \tilde{X} \) is an infinitesimal affine transformation of \( T(M) \) with \( H \nabla \). Since \( \tilde{X} \) has the components \( \left( \frac{X^h}{\partial X^k} \right) \), it follows that \( \tilde{X} \cdot \tilde{X} \) is a vertical infinitesimal affine transformation in \( T(M) \) with \( H \nabla \). Thus we have,

Theorem 5. If \( \tilde{X} \) is a fibre-preserving infinitesimal affine transformation of \( T(M) \) with the lift \( H \nabla \), then \( \tilde{X} = \tilde{X} + \gamma D + \gamma C \), where \( D \) and \( C \) are tensor fields of type \((1,0)\) and \((1,1)\), respectively, satisfying conditions (a), (b) and (c) of Theorem 1.

4. INFINITESIMAL ISOMETRY WITH \textsuperscript{H} g

A vector field \( X \in \mathcal{S}_0(M) \) is said to be an infinitesimal isometry or a Killing vector field of a Riemannian manifold with metric \( g \), if \( L_X g = 0 [4] \). In terms of components \( g_{ij} \) of \( g \), \( X \) is infinitesimal isometry if and only if

\[
L_X g_{ij} = X^a \nabla_a g_{ij} + g_{ai} \nabla_i X^a + g_{aj} \nabla_j X^a = \nabla_j X_i + \nabla_i X_j
\]

\( X^a \) being components of \( X \), where \( \nabla \) is the Riemannian connection of the metric \( g \).

Let \( \tilde{X} \) be vector field in \( T(M) \) and \( \tilde{X}^a = \left( \frac{\tilde{X}^h}{\tilde{X}^k} \right) \) its components with respect to induced coordinates. Then the covariant derivative \( H \nabla \tilde{X} \) has components
\[ ^h \nabla_j \tilde{X}^j = \partial_j X^j + ^h \Gamma^j_{ij} \tilde{X}^i \]  
(22)

\[ ^s \Gamma^j_{ij} \] being given by (3), with respect to induced coordinates.

We now consider a vector field \( X \in \mathcal{S}_0^1(M_n) \), then its vertical lift \( ^v X \in \mathcal{S}_0^1(T(M_n)) \), complete lift \( ^c X \in \mathcal{S}_0^1(T(M_n)) \) and horizontal lift \( ^h X \in \mathcal{S}_0^1(T(M_n)) \) have respectively components of the form

\[
^v X = \begin{pmatrix} 0 \\ x^h \\ \frac{\partial x^h}{\partial x^j} \end{pmatrix}, \quad ^c X = \begin{pmatrix} x^h \\ -\Gamma^h_i x^i \end{pmatrix}, \quad ^h X = \begin{pmatrix} x^h \\ \nabla^h_i x^i \end{pmatrix}
\]  
(23)

with respect to the induced coordinates in \( T(M_n) \), where \( \Gamma^h_i x^i = y^j \Gamma^h_{ij} x^j \).

We now compute the Lie derivatives of the metric \( ^h g \) with respect to \( ^v X \), \( ^c X \) and \( ^h X \), by means of (3) and (23). The Lie derivatives of \( ^h g \) with respect to \( ^v X \), \( ^c X \) and \( ^h X \) have respectively components

\[
\begin{align*}
L^v_X \, ^h g &= (^h \nabla_i ^v X^j + ^h \nabla_j ^v X^i) = \begin{pmatrix} 0 \\ \nabla_j X^i + \nabla_i X^j \\ 0 \end{pmatrix} \\
L^c_X \, ^h g &= (^h \nabla_i ^c X^j + ^h \nabla_j ^c X^i) = \begin{pmatrix} \nabla_j X^i + \nabla_i X^j \\ \partial_j (\nabla_i X^j + \nabla_i X^i) - \partial^i (R^j_ik + R^ik_j)X^k \\ 0 \end{pmatrix} \\
L^h_X \, ^h g &= (^h \nabla_i ^h X^j + ^h \nabla_j ^h X^i) = \begin{pmatrix} \nabla_j X^i + \nabla_i X^j \\ -\Gamma^j_ik \nabla_k X^h - \Gamma^h_ik \nabla_j X^h \nabla_j X^i + \nabla_i X^i \\ 0 \end{pmatrix}
\end{align*}
\]  
(24)

Taking account of the fact that \( \nabla_i X^k = 0 \) implies \( R^j_ik X^k = 0 \) and \( R^j_ik X^k = 0 \)

We have,

**Theorem 6.** Necessary and sufficient conditions in order that

a) Vertical \( ^v X \in \mathcal{S}_0^1(T(M_n)) \)

b) Complete \( ^c X \in \mathcal{S}_0^1(T(M_n)) \)

c) Horizontal \( ^h X \in \mathcal{S}_0^1(T(M_n)) \)

lifts to \( T(M_n) \) with the metric \( ^h g \), of a vector field \( X \) in \( M_n \) be a Killing vector field in \( T(M_n) \) are that

a) \( X \) is infinitesimal isometry in \( M_n \)

b) \( X \) is infinitesimal isometry in \( M_n \) with vanishing covariant derivation in \( M_n \).

c) \( X \) is infinitesimal isometry in \( M_n \) with vanishing covariant derivation in \( M_n \).

Let \( X \) and \( Y \) be vector fields in \( M_n \). If \( X \) and \( Y \) are Killing vector fields in \( M_n \), from the definition of Killing vector field, then we have,
\[ L_{[x,y]}g = L_x(L_y g) - L_y(L_x g) = 0 \]

i.e. \([X, Y]\) is infinitesimal isometry in \(M_n\) [3].

We now denote by \(A_X Y\) the tensor field of type (1,1), \(X\) and \(Y\) being two given elements of \(\mathfrak{S}_0^1(M_n)\), defined by

\[ (A_X Y)Z = (L_X \nabla)(Y, Z) = [L_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \]

for any \(Z \in \mathfrak{S}_0^1(M_n)\). Then we have,

\[ \left[ {}^vX, {}^vY \right] = {}^v[X, Y], \left[ {}^cX, {}^cY \right] = {}^c[X, Y], \left[ {}^cX, {}^hY \right] = {}^h[X, Y] - \gamma(A_X Y) \cdot (27) \]

where \(\gamma(A_X Y) \in \mathfrak{S}_0^1(M_n)\), which has the components of the form \(0_{y^k(A_X Y)}\). An infinitesimal transformation defined by vector field \(X \in \mathfrak{S}_0^1(M_n)\) is said to be infinitesimal transformation with affine connection \(\nabla\), if \(L_X \nabla = 0\). Then, from (26) and (27)

\[ \left[ {}^cX, {}^hY \right] = {}^h[X, Y] \]

We compute the Lie derivatives of the metric \(h g\) with respect to \(^v[X, Y]\) and \(^c[X, Y]\)

\[ L^h_{[x,y]} g = L^h_{[x,y]} g = L_{[x,y]}(L_{c,y} h g) - L_{c,y} (L_{x} h g) \]

\[ L^h_{[x,y]} g = L^h_{[x,y]} g = L_{[x,y]}(L_{c,y} h g) - L_{c,y} (L_{x} h g) \]

from (24) and (29), we get,

**Theorem 7.** Sufficient conditions in order that the vertical, complete lifts of a vector field \([X, Y]\) in \(M_n\) to \(T(M_n)\) be a infinitesimal isometry with metric \(h g\) are that \(X\) and \(Y\) is a infinitesimal isometry in \(M_n\) with vanishing their covariant derivations in \(M_n\). Let \(X\) be infinitesimal affine transformation in \(M_n\). From (25) and (29), we have

\[ L^h_{[x,y]} g = L^h_{[x,y]} g = L_{[x,y]}(L_{c,y} h g) - L_{c,y} (L_{x} h g) \]

from (24) and (30), we get,

**Theorem 8.** Sufficient conditions in order that the horizontal lift of a vector field \([X, Y]\) in \(M_n\) to \(T(M_n)\) be a infinitesimal isometry with metric \(h g\) are that \(X\) and \(Y\) are infinitesimal isometry with vanishing their covariant derivations in \(M_n\).
REFERENCES