On Fixed Point Theory for Generalized Contractions in Cone Metric Spaces Via Scalarizing

Parastoo Zangenehmehr[a], Ali Farajzadesh [b], Sayed Mansour Vaezpour [c]
[a] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
[b] Department of Mathematics, Razi University, Kermanshah, 67149, Iran.
[c] Department of Mathematics, Amirkabir University of Technology, 15916 34311, Tehran, Iran.
*Author for correspondence; e-mail: zangeneh_p@yahoo.com, farajzadehali@gmail.com, ali-razi.ac.ir, vaez@aut.ac.ir

ABSTRACT

In this paper, some fixed point theorems for generalized contractions in cone metric spaces are provided. The normal condition on the underlying cone is omitted. Moreover, the equivalency between the ordered boundedness and topologically boundedness, without using normality on the cone, for a subset of an ordered topological vector space is presented. The results of this article can be considered as the extension of [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., 136(5) (2008), 1861-1869], [M. Kikkawa and T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, Nonlinear Anal., 69(9) (2008), 2942-2949] and [A.P. Farajzadeh, A. Amini-Harandi, D. Baleanu, Fixed point theory for generalized contractions in cone metric spaces, Commun. Nonlinear. Sci. Numer. Simulat. 17(2)(2012) 708-712].

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1. INTRODUCTION

Investigation of K-metric spaces (also known as cone metric spaces) was introduced by several Russian authors in the middle of 20th century [19]. Ordered normed spaces and cones have applications in applied mathematics, for instance in using Newton’s approximation method [17] They differ from usual metric spaces in the fact that the values of distance functions are not positive real numbers, but elements of a cone in some normed spaces [10] or topological vector spaces [7]. L.G. Huang and X. Zhang [10], re-introduced K-metric spaces by the new name cone metric spaces (from now on we will use this name) and also they went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. They proved some fixed point theorems for contractive mappings in cone metric spaces. Since then, the fixed point theory for mappings in cone metric spaces has become a subject of interest till now by many authors [1, 2, 5, 10, 11, 12, 15, 18] and references therein. Further, there are many fixed point results for generalized contractions in metric spaces which were extended to cone metric spaces.
by several authors in the past, see [1, 2, 10, 11, 12, 15] and references therein. Our aim in this paper is to prove some fixed point results for generalized contractions in cone metric spaces with a cone which is not necessarily normal. Moreover, the equivalency between the ordered boundedness and topologically boundedness, without using normality on the cone, for a subset of an ordered topological vector space is presented. The results of this article can be considered as the extension of [8, 13, 16]. In the rest of this section we recall some definitions and preliminary results which we need in the sequel.

Let $E$ be a vector space with zero vector $\theta$. By a cone $P \neq \{\theta\}$ we understand a convex subset of $E$ such that $\lambda P \subseteq P$ for all $\lambda \geq 0$ and $P \cap -P = \{\theta\}$. Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to $P$ by $x \preceq y$ if and only if $y-x \in P$. We shall write $x \prec y$ to indicate that but $x \preceq y$, while $x \succ y$ will stand for $y-x \in P$ (where int $P$ denotes the interior of $P$) if $P$ has nonempty interior. In the following we always suppose that $E$ is a real topological vector space (briefly, t.v.s), unless the contrary is explicitly stated, with zero vector $\theta$. A cone with $P \neq \emptyset$, $e \in \text{int} P$ and $\preceq$ a partial ordering induced by $P$. The cone $P$ is called regular when $\|x\| = \inf\{r \in \mathbb{R}; x \prec re\}$ and $\sup \{r \in \mathbb{R}; x \succ re\}$.

By a cone metric space we mean an ordered pair $(X, d)$ where $X$ is any nonempty set and $d : X \times X \rightarrow E$ is a mapping, called cone metric, satisfying the following conditions:

1) $d(x, y) = 0 \iff x = y$;
2) $d(x, y) = d(y, x), \forall x, y \in X$;
3) $d(x, z) \preceq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

It is easy to see that if $(X, d)$ is a cone metric space then the family $\{B(x_n) \mid (x_n) \in X \times \text{int} P\}$ is a basis for a topology on $X$ where $B(x_n) = \{y \in X : d(x_n, y) \preceq e\}$ and $e \in \text{int} P$.

Hence the sequence $\{x_n\}$ of cone metric space $X$ converges to $x \in X$ if for any $e \in \text{int} P$ there exists natural number $m$ such that $d(x_n, x) \preceq e$ for all $n > m$. Similarly we can define a Cauchy sequence. The cone metric space $(X, d)$ is called complete if every Cauchy sequence of it is convergent to a point in $X$. The next definition was first introduced in [9] in order to apply in optimization theory and then was used for equilibrium problems in [4].

**Definition 1.1.** Let $E$ be a real topological vector space and $P$ a cone of $E$ with $e \in \text{int} P$. The nonlinear scalarization function $\xi : E \rightarrow \mathbb{R}$ (the set of real numbers) is defined as follows:

$\xi (y) = \inf \{r \in \mathbb{R}; y \in Re-P\}$

= $\inf \{r \in \mathbb{R}; y \preceq re\}$

**Lemma 1.2.** ([7, 9]) For each $r \in \mathbb{R}$ and $y \in E$, the following statements are satisfied:

1) $\xi (y) \leq r \iff y \in \text{re-P} \iff y \preceq re$;
2) $\xi (y) < r \iff y \in \text{re-int} P \iff y \prec re$;
3) if $y \prec y$, then $\xi (y) \leq \xi (y)$;
4) $\xi (\cdot)$ is continuous, positively homogeneous (that is, $\xi (tx) = t\xi (x)$, for all $x \in E$ and nonnegative number $t$) and subadditive on $E$ (that is, $\xi (x+y) \leq \xi (x) + \xi (y), \forall x, y \in E$);
5) $\xi (x) - \xi (y) \leq \xi (x-y)$, for all $x, y \in E$.

The next result plays a crucial role in the next section.

**Lemma 1.3.** ([7]) Let $E$ be a real topological vector space, $P$ a cone of $E$ with $e \in \text{int} P$ and $d : X \times X \rightarrow E$ a cone metric on $X$. Then $\xi_{od} : X \times X \rightarrow \mathbb{R}$ is a metric on $X$, where $\xi_{od}$ denotes the composition of $\xi$ with $d$, that is $\xi_{od}(x, y) = \xi (d(x, y))$, for all $(x, y) \in X \times X$. Furthermore, if $(X, d)$ is complete cone metric
space then \((X, \xi, od)\) is also complete metric space.

2. Main results

In this section we present some fixed point theorems for generalized contractions in cone metric spaces are provided. The normal condition on the underlying cone is omitted. Further, equivalency between the ordered boundedness and topologically boundedness, without using normality on the cone, for a subset of an ordered topological vector space is presented and under sufficient condition in order to close the fixed point is investigated.

In 2008 Suzuki[16] proved the following fixed point theorem for generalized contractions in complete metric spaces.

**Theorem 2.1.** Let \((X, \rho)\) be a complete metric space and let \(T:X \to X\) be a map. Define a non-increasing function \(\gamma:[0,1) \to [\frac{1}{3},1]\) by

\[
\gamma(r) = \begin{cases} 
1, & \text{if } 0 \leq r \leq \frac{\sqrt{5} - 1}{2}; \\
(1-r)r^{-2}, & \text{if } \frac{\sqrt{5} - 1}{2} \leq r < \frac{1}{2}; \\
(1+r)^{-1}, & \text{if } \frac{1}{2} \leq r < 1.
\end{cases}
\]

Assume that there exists \(r \in [0;1)\) such that \(\gamma(r)\rho(x,T(x)) \leqslant \rho(y,T(y))\) implies \(\rho(T(x); T(y)) \leqslant \gamma(r)\rho(x,y)\) for all \(x, y \in X\). Then there exists a unique fixed point \(z\) of \(T\). Moreover, \(\lim_{n \to \infty} T^n x = z\) for all \(x \in X\).

**Proof.** It is clear from Lemma 1.3 that \((X, \xi, od)\) is a complete metric space. Let \(\gamma(r)\xi, od(x,T(x)) \leqslant \xi, od(x,y)\). Then, by the properties of \(\xi\) that is Lemma 1.2, we have

\[
0 \leq \xi, od(x,y) - \gamma(r)\xi, od(x,T(x)) = s \xi, (d(x,y), y) - s \xi, (d(x,y), T(x)) \leq \xi, (d(x,y), \gamma(r)\xi, d(x,y)).
\]

So it follows from Lemma 1.2 (2) that

\[
\omega (x,y) \leq \gamma(r)\xi, d(x,y)
\]

and then

\[
\omega (x,y) \leq \gamma(r)\xi, d(x,y)
\]

Hence it follows from Lemma 1.2 (3) that

\[
\xi, (d(x,y), y) \leq \gamma(r)\xi, d(x,y)
\]

Now the result follows from Theorem 2.1. □

The following example satisfies in Theorem 2.2.

**Example 2.3.** Define a complete cone metric space \(X\) with cone \(P = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}\) by \(X = \{(0,0), (4,0), (0,4), (4,5), (5,4)\}\) and its metric \(d\) by

\[
d((x_1,x_2),(y_1,y_2)) = \begin{cases} 
|x_1 - y_1| + |x_2 - y_2| & \text{if } x_1 \leq x_2, \\
|y_1 - y_2| + |x_2 - y_2| & \text{if } x_1 > x_2.
\end{cases}
\]

Define a mapping \(T\) on \(X\) by

\[
T(x_1,x_2) = \begin{cases} 
(x_1,0) & \text{if } x_1 \leq x_2, \\
(0,x_2) & \text{if } x_1 > x_2.
\end{cases}
\]

Then \(T\) satisfies the assumption in Theorem 2.2, and then there exists a unique fixed point. Moreover, \(\lim_{n \to \infty} T^n x = z\) for all \(x \in X\). (it is clear that in this example \(z = (0,0)\)).

Let \((X, \rho)\) be a metric space. We denote by \(CB(X)\) the family of all nonempty closed bounded subsets of \(X\). Let \(H_e (., .)\) be the Hausdorff metric with respect to \(\rho\), i.e.,

\[
H_e (A,B) = \max \{ \sup_{x \in A} \rho(x,B), \sup_{y \in B} \rho(y,A) \},
\]

for all \(A,B \in CB(X)\), where \(\rho(x,B) = \inf_{y \in B} \rho(x,y)\).

Kikkawa and Suzuki [13] proved the following generalization of Nadler fixed point theorem for set valued mappings. [14].
Theorem 2.4. Let \( \eta[0,1) \rightarrow ([0,1]) \) be defined by \( \eta(r) = \frac{1}{r^2} \), \((X, d)\) a complete metric space, and let \( T: X \rightarrow X \) be a mapping from \( X \) into \( CB(X) \). Assume that there exists \( r \in [0,1] \) such that, for all \( x, y \in X \),
\[
\eta(r) d(x, Ty) \\leq g(x, y) \quad \text{implies} \quad H \left( (T_x, Ty) \right) \\leq r g(x, y). \quad (2.3)
\]
Then there exists \( z \in X \) such that \( z \in Tz \). In the next we want to obtain cone metric version of Theorem 2.4. Before that we need the following definition and lemma. Let \((X, d)\) be a cone metric space and let \( E \) be a real topological vector space and \( P \) a cone of \( E \). A set \( D \subseteq X \) is called \textit{bounded} if the set \( \{d(x, y) : x, y \in D\} \) is topologically bounded subset of \( E \) \([3, 6]\), that is \( \lim_{n \to \infty} \frac{1}{n} d(x_n, y_n) = 0 \) for any \( \{(x_n, y_n)\} \subseteq D \times D \). We denote by \( CB(X) \) the family of all nonempty closed bounded subsets of \( X \).

The following theorem establishes a link between topologically boundedness and order boundedness. Furthermore, it is a topological vector space version of Lemma 2.6 in [8], without assuming the normal condition on the cone and considering boundedness for only subsets of the cone, and repair the proof given for it by providing a new proof.

Lemma 2.5. Let \( E \) be a real topological vector space and \( P \) a cone of \( E \) with \( \text{int} P \neq \emptyset \). Then \( C \subseteq E \) is topologically bounded if and only if \( C \) is order bounded.

Proof. Let \( C \subseteq E \) be topologically bounded and \( r \in \text{int} P \). On the contrary if \( C \) is not order bounded then for each natural number \( k \) there exists \( x_k \in C \) such that \( x_k \notin kE \). Then \( kE \supseteq x_k \in E \setminus \text{int} P \) (the complement \( \text{int} P \) in \( E \)) and so \( e^{-\frac{X_n}{k}} \notin E \setminus P \). Hence, by letting \( k \to \infty \), we get \( e \notin E \setminus P \) (note \( E \setminus P \) is closed and \( \lim_{k \to \infty} e^{-\frac{X_n}{k}} = 0 \)) which is a contradiction.

 Conversely, let there exist \( z \in E \) such that \( z \leq \zeta \) for all \( x \in C \). If \( C \) is not topologically bounded then there exists sequence \( \{x_n\} \) of \( C \) such that for each neighborhood \( U \) of \( \theta \), especially \( U = kE \text{-int} P \), for each natural \( k \), there exists \( x_k \in C \) such that \( x_k \notin kE \text{-int} P \). So \( e^{-\frac{X_n}{k}} \notin \text{int} P \) and \( \{e^{-\frac{X_n}{k}}\} \in P \) (note \( x \leq \zeta \), for all \( x \in C \)) imply that
\[
e^{-\frac{X_n}{k}} = \left( e^{-\frac{X_n}{k}} \right) + \left( \frac{X_n}{k} \right)
\in \left( E \setminus P \right) \subseteq \left( E \setminus \text{int} P \right),
\]
and so, since \( E \setminus P \) is closed and \( e^{-\frac{X_n}{k}} \) converges to zero, when \( k \to \infty \) we deduce that \( e \in E \setminus \text{int} P \) which is impossible because \( e \in \text{int} P \).

This completes the proof.

The authors of [8], see page 711 before Theorem 2.7 of [8], defined the Hausdorff metric \( H \) on \( CB(X) \), for cone metric space \((X, d)\) by
\[
H \left( A, B \right) = \sup \left\{ \sup_{a \in A} d(a, B), \sup_{y \in B} d(y, A) \right\}
\]
for all \( A, B \in CB(X) \), where \( d(x, B) = \inf_{y \in B} d(x, y) \), and they applied \( H \) in Theorem 2.7. It is worth noting that in order to define the Hausdorff metric for a cone metric it is necessary that the cone should be strongly minihedral, that is for any nonempty subset \( E \) which is bounded above with respect to the ordering induced by \( P \) has least upper bound. The following theorem is an extension of Theorem 2.4 to the cone metric spaces and Theorem 2.7 in [8] for the cone which is not necessarily normal and strongly minihedral.

Theorem 2.6. Let the function \( \eta \) be as given in Theorem 2.4, \((X, d)\) a complete cone metric space, and \( T: X \rightarrow X \) a mapping from \( X \) into \( CB(X) \). Assume that there exists \( r \in [0,1) \) such that
\[
\eta(r) d(x, Ty) \Rightarrow 
\quad \text{implies} \quad H \left( (T_x, Ty) \right) \\leq \text{rd}(x, y). \quad (2.4)
\]
For all \( x, y \in X \). Then there exists \( z \in X \) such that \( z \leq \zeta \).

Proof. Let \( g(x, y) = \zeta \left( d(x, y) \right) \). By Lemma 1.3, \( \varrho \).
is a complete metric on \( X \) and it follows from Lemma 1.2 that if \( A \in CB(X) \) (that is closed with respect to \( d \)) then it is bounded closed with respect to \( g(x,y) \). Then \((CB(X), H_\varepsilon)\) is a complete metric space. The conclusion follows from Theorem 2.4, if we show that the relation (2.3) holds. Assume that
\[
\eta(r)\rho(x,Tx) = \eta(r)\xi_e(\rho(x,T(x))) \\
\leq \rho(x,y) \\
= \xi_e(\rho(x,y)) \ 	ext{for} \ x, y \in X. \quad (2.5)
\]
Then by applying Lemma 1.2 we get
\[
0 \geq \xi_e(\rho(x,y)) - \eta(r)\xi_e(\rho(x,T(x))) \\
\leq \xi_e(\rho(x,y) - \eta(r)\rho(x,T(x))),
\]
and so
\[
d(x,y)\eta(r)d(x,T(x)) \in \text{intP}.
\]
This means
\[
d(x,y) \xi \leq \eta(r)d(x,T(x)),
\]
and so from (2.4) we have
\[
H_\varepsilon(Tx; Ty) \leq rd(x,y)
\]
and then since \( \xi \) is monotone (Lemma 1.2 (3)) we get
\[
H_\varepsilon(Tx; Ty) \leq \rho(x,y)
\]
(Noe that for every \( x \in B \) and \( x \in A \) \((A,B \in CB(X))\) we have \( d(x,y) \leq \sup d(x,B) \)
and then since \( \xi \) is monotone we get
\[
\xi (d(x,y)) \leq \xi(\sup d(x,B)) \leq \xi(rd(x,y)) = r\xi(d(x,y))
\]
and so
\[
\sup_{y \in B} \xi(y)(d(x,y)) \leq r\xi(d(x,y)).
\]
Now the result follows from Theorem 2.4. □

By reviewing the proof of Theorem 2.6, one can deduce that the following corollary.

**Corollary 2.7.** Let \((X,d)\) be a complete cone metric space and \( T:X \to CB(X) \) a mapping from \( X \) into \( CB(X) \). Assume that there exists \( r \in [0,1] \) such that
\[
H_\varepsilon(Tx; Ty) \leq r\xi(d(x,y)) \quad (**) \text{for all} \ x,y \in X.
\]
Then there exists \( \varepsilon \in X \) such that \( \varepsilon \in T\varepsilon \).

3. **Conclusion**

We do not use normality and strongly minihedrality assumptions due to the relation (**) More precisely, if \( A \) is a bounded subset of \( X \) then the set \( \{\xi, (d(x,y)): x,y \in A\} \) is a bounded subset of the real line and so if \( Tx \) and \( Ty \) \((x,y \in X)\) are bounded subset of \( X \) then \( H_\varepsilon(Tx; Ty) \) is well-defined. So Corollary 2.7 improves Corollary 2.8 in [8] by relaxing normality and strongly minihedral on the cone. Also, it is a generalization of both Nadler's fixed point Theorem [14] and [[10], Theorem1]. Theorem 2.6 is an extension of Theorem 2.4 to the cone metric spaces and Theorem 2.7 in [8] for the cone which is not necessarily normal and strongly minihedral.

**References**


