Research Article

A Characterization of Groups Whose Lattices of Subgroups are \( n-M_{p+1} \) Chains for All Primes \( p \)

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Abstract

Whitman, P.M. and Birkhoff, G. answered a well-known open question that for each lattice \( L \) there exists a group \( G \) such that \( L \) can be embedded into the lattice \( \text{Sub}(G) \) of all subgroups of \( G \). Gratzer, G. has characterized that \( G \) is a finite cyclic group if and only if \( \text{Sub}(G) \) is a finite distributive lattice. Ratanaprasert, C. and Chantasartrassmee, A. extended a similar result to a subclass of modular lattices \( M_m \) by characterizing all integers \( m \geq 3 \) such that there exists a group \( G \) whose \( \text{Sub}(G) \) is isomorphic to \( M_m \) and also have characterized all groups \( G \) whose \( \text{Sub}(G) \) is isomorphic to \( M_m \) for some integers \( m \). On the other hand, a very well-known open question in Group Theory asked for the number of all subgroups of a group. In this paper, we consider the extension of the subclass \( M_m \) for all integers \( m \geq 3 \) of modular lattices, the class of \( n-M_{p+1} \) chains for all primes \( p \), and all \( n \geq 1 \) and characterized all groups \( G \) whose \( \text{Sub}(G) \) is an \( n-M_{p+1} \) chain. It happens that \( G \) is a group whose \( \text{Sub}(G) \) is an \( n-M_{p+1} \) chain if and only if \( G \) is an abelian \( p \)-group of the form \( Z_r \times Z_r \). Moreover, we can tell numbers of all subgroups of order \( p_i \) for each \( 1 \leq i \leq n \) of the special class of \( p \)-groups.

Key Words: Modular lattice; Lattice of subgroups; \( p \)-group

Introduction

A lattice \( L \) is a non-empty ordered set in which each pair of elements \( a, b \) of \( L \) has the least upper bound denoted by \( a \lor b \) and the greatest lower bound denoted by \( a \land b \). Whitman, P.M. (1946) proved that for each lattice \( L \) there exists a set \( X \) such that \( L \) can be embedded into the lattice of all equivalence relations on \( X \). One can show that the set \( \text{Sub}(G) \) of all subgroups of a group \( G \) forms a lattice in which \( H \lor K = \langle H \cup K \rangle \) and \( H \land K = H \cap K \) for each pair of elements \( H, K \) of \( \text{Sub}(G) \). We call \( \text{Sub}(G) \), the lattice of subgroups. Birkhoff, G. (1967) proved that every lattice of all equivalence relations on a set \( X \) is isomorphic to the lattice \( \text{Sub}(G) \) of a group \( G \). These results answered a well-known open question that for each lattice \( L \) whether there exists a group \( G \) such that \( L \) can be embedded into \( \text{Sub}(G) \).

A lattice \( L \) is said to be distributive if it satisfies the distributive law; that is, \( (a \land b) \lor (a \lor c) = a \lor (b \land c) \) for all \( a, b, c \in L \). Zembery, I. (1973) answered...
the open question in a special class of lattices by proving that every finite distributive lattice can be embedded into \( \text{Sub}(G) \) for some abelian group \( G \). Further, Gratzer, G. (1978) has characterized that \( G \) is a finite cyclic group if and only if \( \text{Sub}(G) \) is a finite distributive lattice; and he also proved that \( \text{Sub}(G) \) of a finite cyclic group \( G \) is isomorphic to a product of finite chains. We can conclude that for each finite distributive lattice \( L \) there exists a finite cyclic group \( G \) such that \( L \) can be embedded into \( \text{Sub}(G) \). A lattice \( L \) is said to be modular if it satisfies the modular law; that is, \( a \geq c \) implies that \( a \land (b \lor c) = (a \land b) \lor c \) for all \( a,b,c \in L \). It is well-known that if \( L \) is distributive then \( L \) is modular. Let \( m \geq 3 \) be a positive integer and let \( M_m \) be the set \( \{0,1,a_1,a_2,\ldots,a_m\} \) satisfying \( 0 \leq x \leq 1 \) for all \( x \in M_m \) and has no other comparabilities. It is obvious that \( M_m \) is a finite modular lattice which is not distributive for each \( m \geq 3 \). It is also proved by Fraleigh, J. B. (1982) that if \( G \) is a group whose \( \text{Sub}(G) \) is isomorphic to \( M_m \) for some \( m \geq 3 \) then \( G \) is not cyclic. It is known that if \( G \) is an abelian group then \( \text{Sub}(G) \) is modular; but the converse is not always true; for instance, \( \text{Sub}(D_3) \) the set of all subgroups of the dihedral group \( D_3 \) is isomorphic to \( M_4 \). Ratanaprasert, C. and Chantasartrassmee, A. (2004) have characterized all groups \( G \) whose \( \text{Sub}(G) \) is isomorphic to \( M_m \) for some \( m \geq 3 \). We proved the following theorems.

**Theorem 1.1**: Let \( m \geq 3 \) be a positive integer. Then there is a group \( G \) whose \( \text{Sub}(G) \) is isomorphic to \( M_m \) if and only if \( m = p+1 \) for some prime \( p \).

**Theorem 1.2**: Let \( G \) be a group. Then \( \text{Sub}(G) \) is isomorphic to \( M_3 \) if and only if \( G \) is isomorphic to \( Z_2 \times Z_2 \).

**Theorem 1.3**: Let \( G \) be a group and \( p \) be a prime number. Then \( \text{Sub}(G) \) is isomorphic to \( M_{p+1} \) if and only if either \( G \) is isomorphic to \( Z_p \times Z_p \) or \( G \) is a non-abelian group of order \( pq \), where \( q \) is a prime number with \( q \) divides \( p-1 \), generated by elements \( c, d \) such that \( c^p = d^q = e \), where \( e \) denotes the identity of \( G \) and \( dc = c^sd \) where \( s \) is not congruence to 1 modulo \( p \) and \( s^q \equiv 1 \pmod{p} \).

**Corollary 1.4**: Let \( G \) be a non-abelian group whose \( \text{Sub}(G) \) is isomorphic to \( M_{p+1} \) for some prime \( p \). Then (i) \( p \) is an odd prime and (ii) \( G \) is of order \( pq \) where \( q \) is a prime number with \( q \) divides \( p-1 \) and \( G \) contains exactly one subgroup of order \( p \) and \( p \) subgroups of order \( q \).

**Groups whose lattices of subgroups are \( n\text{-M}_3 \) chains**

By the Structure Theorems for Finite Abelian Groups and Theorem 1.2, we look for the diagram of the lattice \( \text{Sub}(Z_2 \times Z_2) \) of all subgroups of the abelian \( p \)-group \( Z_2 \times Z_2 \) where \( Z_2 = \{0, 1, 2, 3\} \) be the (additive) group of integers modulo 4. One can see that all subgroups of the direct product \( Z_2 \times Z_2 = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\} \) are \( a_{03} := \{(0,0)\} = <(0,0)> \), \( a_{11} := \{(0,0), (0,1)\} = <(0,1)> \), \( a_{12} := \{(0,0), (2,1)\} = <(2,1)> \), \( a_{13} := \{(0,0), (2,0)\} = <(2,0)> \), \( a_{21} := \{(0,0), (0,1), (2,0), (2,1)\} = <(0,1), (2,0)> \), \( a_{22} := \{(0,0), (1,1), (2,0), (3,1)\} = <(1,1)> \), \( a_{23} := \{(0,0), (1,0), (2,0), (2,1)\} = <(1,0)> \) and \( a_{31} := Z_2 \times Z_2 = <(1,0), (0,1)\> \); and the diagram of the lattice \( \text{Sub}(Z_2 \times Z_2) \) is shown in Figure 1(a). For general case, we have the following proposition.

**Proposition 2.1**: For each integer \( n \geq 2 \), all subgroups of \( Z_2 \times Z_2 \) are (a) \( <(1,0)\) \), (b) \( <(0,1)\) \), (c) \( <(1,1)\) \), (d) \( <(2,0)\) \), (e) a subgroup of \( <(2,0)\) \), \( <(0,1)\) \).

**Proof**: Let \( T \) be a subgroup of \( Z_2 \times Z_2 \) and for \( i \in \{1,2\} \) let \( p_i \) be the projection maps of \( Z_2 \times Z_2 \) on \( Z_2 \) and \( Z_2 \), respectively. Then each \( p_i \) for \( i \in \{1,2\} \) is a homomorphism; hence, \( p_1(T) \) and \( p_2(T) \) are
subgroups of $Z_{2^n}$ and $Z_2$, respectively. If $|T| = 2^{n+1}$ then $T = Z_{2^n} \times Z_2 = \langle(1,0), (0,1)\rangle$. Now, we consider the case $|T| = 2^n$. If $p_2(T) = \{0\}$ then $p_1(T) = Z_{2^n}$; hence, $T = Z_{2^n} \times \{0\} = \langle(1,0)\rangle$. We assume that $p_2(T) = \{0,1\} = Z_2$. If $1 \in p_1(T)$ then $(1,1) \in T$; so, $T$ is a cyclic subgroup of $< (2,0), (0,1)>$. We consider the case $a_{pq} \neq a_{rs} \neq a_{uv}$.

We will generalize the lattice in Figure 1(b) in the following proposition.

**Proposition 2.2**: Let $n$ be a positive integer and let $\leq$ be the usual order on the set $\mathbb{Z}^+ \cup \{0\}$ of all nonnegative integers. If $L := \{a_j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq 3 \} \cup \{a_{ij}, a_{(n+1)i}\}$ and $\leq L \times L$ is defined by $a_{ij} \leq a_{ij} \leq a_{ij}$ for all $0 \leq v < i < u \leq n+1$ and all $1 \leq j \leq 3$ and there are no other comparabilities, then $L = (L; \leq)$ is a lattice.

**Proof**: It is obvious from the definition of $\leq$ that $\leq$ is reflexive. Let $x, y \in L$ satisfy $x \leq y$ and $y \leq x$. Then there are integers $p, q, r, s \in \{0, 1, 2, ..., n+1\}$ such that $x = a_{pq}$ and $y = a_{rs}$. If $q = 3$ and since $a_{rs} = x \leq y = a_{pq}$, we have $s = 3$; but $p \neq r$ implies by the definition of $\leq$ that $r \neq p$ and $p \neq r$ which contradicts to the trichotomy law for $\leq$; hence, $p = r$; and so, $x = a_{pq} = a_{rs} = y$. If $q = 2$ then $a_{pq} = x \leq y = a_{rs}$ implies that $s = 1$ or $s = 2$; but $s = 1$ implies $a_{rl} = y \leq x = a_{p2}$, which contradicts to the definition of $\leq$; so, $s = 2 = q$. Also, $p \neq r$ implies a similar contradiction as above; hence, $p = r$. Therefore, $x = y$. If $q = 1$, then $a_{pq} = x \leq y = a_{rs}$ which shows $s = 1$ and $p \leq r$. Now, $a_{rs} = y \leq x = a_{pq}$ implies that $r \neq s$. So, $p = r$. Hence, $x = y$. In any cases, $x = y$ which shows that $\leq$ is anti-symmetric.

Now, let $x, y, z \in L$ satisfy $x \leq y$ and $y \leq z$. Then there are integers $p, q, r, s, u, v \in \{0, 1, ..., n+1\}$ such that $x = a_{pq}$, $y = a_{rs}$, and $z = a_{uv}$; so, $a_{pq} \leq a_{rs}$ and $a_{rs} \leq a_{uv}$. Since $a_{pq} = a_{rs}$ or $a_{rs} = a_{uv}$ implies that $x \leq z$, we consider the case $a_{pq} \neq a_{rs}$ and $a_{rs} \neq a_{uv}$ which implies by the definition of $\leq$ that $p \neq r$ and $r \neq u$; so, $p \neq u$. If $q = 3$ then $x = a_{pq} \leq a_{uv} = z$. And if $q = 2$ then $s = 1$; and so $a_{pq} \leq a_{uv}$ since $p \neq u$ implies that $v = 1$ and...
p < *u. Finally, if q = 1 then s = v = 1; and so, a_\text{pq} \leq a_{uv} follows from p < * u. Hence, in which cases, x \leq z. Therefore, \leq is transitive.

To show that L is a lattice, let x, y \in L. If x \leq y or y \leq x then x \lor y and x \land y are in the set \{x, y\}. Let x and y be non-comparable. Then there are integers p, q, r, s \in \{0, 1, 2, \ldots, n+1\} such that x = a_{pq} and y = a_{rs}.

We may assume that p \leq r. Then, since a_{pq} and a_{rs} are non-comparable, 1 \leq *p \leq *n and 1 \leq *r \leq *n.

If q = 1 then s \in \{2, 3\}. Since there are no integers c and d with p - 1 < c < *p and r < *d < r + 1, we have a_{(p-1)3} \land a_{p3} = x \leq a_{(r+1)3} and a_{(p-1)3} \land a_{r3} = y \leq a_{(r+1)3} which shows x \land y = a_{(p-1)3} and x \lor y = a_{(r+1)3}.

If q = 2 and r = p then a_{pq} \land a_{pq} = a_{(p-1)3} and a_{pq} \lor a_{pq} = a_{(p+1)3}; but if q = 2 and p < *r then s \in \{2, 3\}; so, x \land y and x \lor y will be as in the case q = 1. And if q = 3 then p = r; so, a_{pq} \leq a_{pq} for all i with p < *i and for all 1 \leq *i \leq 3; so x \land y and x \lor y are as in the case q = 2 and r = p.

**Definition**: The lattice defined as in Proposition 2.2 is called n-M\textsubscript{3} chain.

Figure 1(b) shows the diagram of n-M\textsubscript{3} chain for n \geq 1. For a special case, we note that M\textsubscript{1} is 1-M\textsubscript{3} chain and Theorem 1.2 showed that Sub(Z\textsubscript{2} \times Z\textsubscript{2}) is isomorphic to 1-M\textsubscript{3} chain (which is M\textsubscript{3}). We now prove in general case that Sub(Z\textsubscript{2} \times Z\textsubscript{2}) is isomorphic to n-M\textsubscript{3} chain for each positive integer n.

**Proposition 2.3**: Sub(Z\textsubscript{2} \times Z\textsubscript{2}) is isomorphic to n-M\textsubscript{3} chain for each positive integer n.

**Proof**: We will prove the proposition by mathematical induction. By Theorem 1.2, Sub(Z\textsubscript{2} \times Z\textsubscript{2}) is isomorphic to 1-M\textsubscript{3} chain. We may assume that k is a positive integer such that Sub(Z\textsubscript{2} \times Z\textsubscript{2}) is isomorphic to k-M\textsubscript{3} chain and we will prove the proposition for k + 1.

By Proposition 2.1, all the subgroups of Z\textsubscript{2} \times Z\textsubscript{2} are \bar{Z} := \langle(1,0),(0,1)\rangle, a := \langle(2,0),(0,1)\rangle, b := \langle(1,1)\rangle, c := \langle(0,1)\rangle or a subgroup of \langle(2,0),(0,1)\rangle. Since \langle(2,0),(0,1)\rangle is isomorphic to Sub(Z\textsubscript{2} \times Z\textsubscript{2}), the induction hypothesis implies that Sub(\langle(2,0),(0,1)\rangle) is isomorphic to k-M\textsubscript{3} chain. It is clear that \{1, a, b, c, d\}, where d = \langle(2,0)\rangle, is isomorphic to M\textsubscript{1}. Hence, Sub(Z\textsubscript{2} \times Z\textsubscript{2}) is isomorphic to (k + 1) - M\textsubscript{3} chain which completes the proof.

Theorem 1.2 and Corollary 1.4(i) also showed that there are no non-abelian groups G such that Sub(G) is isomorphic to n-M\textsubscript{3} chain for all n. We are going to prove in the following theorem that it is also true in the class of n-M\textsubscript{3} chains for all positive integers n.

**Theorem 2.4**: Let G be a group and n \geq 3 be an integer. Then Sub(G) is an n-M\textsubscript{3} chain if and only if G is isomorphic to Z\textsubscript{2} \times Z\textsubscript{2}.

**Proof**: The converse of the theorem follows by Proposition 2.3. Let G be a group whose Sub(G) is an n-M\textsubscript{3} chain. Then G is finite and Theorem 1.2 implies that G cannot be non-abelian; and also, the Structure Theorem of Finite Abelian Group implies that G is of the form Z\textsubscript{p1} \times Z\textsubscript{p2} \times \ldots \times Z\textsubscript{pt}, where p\textsubscript{i} are primes for 1 \leq i \leq t. Since an n-M\textsubscript{3} chain is not distributive, G is not a cyclic group; so, there exists a prime factor p of |G| such that Z\textsubscript{p} \times Z\textsubscript{p} is a subgroup of G. So, Theorem 1.1 told us that Sub(Z\textsubscript{p} \times Z\textsubscript{p}) has at least p + 1 atoms. Hence, Cauchy's Theorem implies that all atoms of Sub(Z\textsubscript{p} \times Z\textsubscript{p}) are atoms of G and there are no other prime q differ from p which is a divisor of |G|. So, p + 1 = 3; that is, p = 2 is the only prime factor of |G|.

If Z\textsubscript{2} \times Z\textsubscript{2} \times Z\textsubscript{2} is a subgroup of G, then one of M\textsubscript{3} in the n-M\textsubscript{3} chain has at least 7 atoms since Z\textsubscript{2} \times Z\textsubscript{2} \times Z\textsubscript{2} contains 7 distinct elements of order 2 which contradicts to the form of an n-M\textsubscript{3} chain that each M\textsubscript{3} in the chain has exactly 3 non-comparable elements. So, G is of the form Z\textsubscript{2} \times Z\textsubscript{2} \times Z\textsubscript{2} for some positive integers n and m. Suppose that n > 1 and m > 1. Then a
subgroup \( Z_{2^n} \times Z_2 \) of \( G \) contains 4 subgroups \( \langle (1,0) \rangle, \langle (0,1) \rangle, \langle (1,1) \rangle \) and \( \langle (2,0),(0,2) \rangle \) of order 4 which are non-comparable in \( \text{Sub}(Z_{2^n} \times Z_2) \) and also are in \( \text{Sub}(G) \). Since \( G \) contains only 3 subgroups of the same order which are non-comparable, we get a contradiction. Hence, \( n = 1 \) or \( m = 1 \). Therefore, \( G \) is \( Z_{2^n} \times Z_2 \) for some positive integers \( n \) which completes the proof.

**Corollary**: A lattice \( L \) is isomorphic to \( Z_{2^n} \times Z_2 \) for some positive integer \( n \) if and only if it is an \( n-M_3 \) chain.

**Groups whose lattices of subgroups are \( n-M_{p+1} \) chains for some odd primes \( p \)**

Let \( p \) be an odd prime number and \( n \) be a positive integer. We will now give the definition of \( n-M_{p+1} \) chains by extending the definition of \( n-M_3 \) chains as follows.

Let \( L := \{ a_i \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq p+1 \} \cup \{ a_{(p+1)i} \mid 1 \leq i \leq n+1 \} \) and \( \leq L \times L \) be defined by \( a_{(p+1)i} \leq a_j \) for all \( 0 \leq i \leq n \text{ and } 1 \leq j \leq p+1 \) and there are no other comparabilities. Then one can repeat the proof in Proposition 2.2 with \( p+1 \) in place of 3 to conclude that \( L = (L; \leq) \) is a lattice which will be called an \( n-M_{p+1} \) chain.

We begin to prove that there is no non-abelian group \( G \) whose \( \text{Sub}(G) \) is isomorphic to an \( n-M_{p+1} \) chain if \( n > 1 \) and \( p > 2 \).

**Proposition 3.1**: If \( G \) is a group whose \( \text{Sub}(G) \) is isomorphic to an \( n-M_{p+1} \) chain for some odd prime \( p \) and some integer \( n > 1 \), then \( G \) is an abelian group of the form \( Z_p \times Z_p \).

**Proof**: Suppose that there is a non-abelian group \( G \) whose \( \text{Sub}(G) \) is isomorphic to an \( n-M_{p+1} \) chain for some integers \( n > 1 \) and primes \( p > 2 \). Then Theorem 1.3 and Corollary 1.4(i) imply that the subgroup \( H := a_{z_1} \) of \( G \) which is the top of the first \( M_{p+1} \) of the \( n-M_{p+1} \) chain must be either \( Z_p \times Z_p \) or a non-abelian group of order \( pq \) where \( q \) is a prime factor of \( p–1 \); hence, the prime \( q \) must be a factor of \( |G| \). If \( H = Z_p \times Z_p \), Cauchy’s Theorem implies that \( |G| \) cannot have other prime factors (except \( p \)); that is, \( G \) is of order \( p^t \) for some positive integer \( t \). Since \( G \) is non-abelian, \( G \) is not \( H = Z_p \times Z_p \); so the subgroup \( a_{z_1} \) of \( G \) is of order \( p^t \). If \( a_{z_1} \) is abelian then \( a_{z_1} \times Z_p \times Z_p \) (as \( a_{z_1} \) cannot be \( Z_p \) since the cyclic group cannot have \( Z_p \times Z_p \) as its subgroup) and \( \text{Sub}(Z_p \times Z_p \times Z_p) \) is not a \( 2-M_p \) chain since it contains \( p^t–1 \) distinct elements of order \( p \) and each generates a subgroup which is an atom of \( \text{Sub}(G) \). So, \( a_{z_1} \) is a non-abelian group of order \( p \) which has elements of order \( p^2 \) and has no elements of order \( p^3 \) (i.e., if all elements of \( a_{z_1} \) are of order \( p \) or there is an element of \( a_{z_1} \) of order \( p^3 \) then either \( \text{Sub}(a_{z_1}) \) contains \( p^3–1 \) atoms which implies that \( \text{Sub}(a_{z_1}) \) is not a \( 2-M_p \) chain or \( a_{z_1} \) is cyclic; in which cases imply a contradiction). Since \( \text{Sub}(a_{z_1}) \) contain \( p+1 \) co-atoms which are subgroups of order \( p^2 \), \( a_{z_1} \) must contain exactly \( (p+1)(p^2–1)+1 = p^3–p \) elements; so, \( p^3–p = p^3 \) which implies that \( p = 0 \) or \( p = 1 \) which contradicts that \( p \) is prime. Therefore, \( G \) is a non-abelian group of order \( pq \) where \( q \) is a prime factor of \( p–1 \); and also, \( p \) and \( q \) are the only prime factors of \( |G| \). If \( n > 1 \), \( \text{Sub}(a_{z_1}) \) contains \( p \) cyclic subgroups of order \( q \) and only one cyclic subgroup of order \( p \) which is \( Z_p \). Since \( \text{Sub}(Z_p \times Z_p) \) is \( M_{p+1} \), the \( a_{z_1} \) in \( \text{Sub}(G) \) must be \( Z_p \) and \( a_{2z_1} \cdots a_{(p+1)z_1} \) are cyclic subgroups of \( Z_p^2 \). So, \( a_{z_1} \) must contain exactly \( p(q(p^2–1)+1) \) elements. By the First Sylow Theorem and \( p, q \) are the only prime factors of \( |G| \), we have \( p+q(p^2–1)+1 \) which implies that \( p = q(p^2–1)+1 \); hence, \( p = q \) or \( p = q(p^2–1)+1 \) which are impossible in both cases. Therefore, \( G \) is an abelian group.

The above argument also shows that there is only
one prime number \( p \) which is a factor of \( |G| \) and \( G \) cannot have \( \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \) as its subgroup; so, \( G \) is of the form \( \mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \mathbb{Z}_p \) for some positive integers \( n \) and \( m \). Hence, a similar proof in Theorem 2.4 implies that \( G \) is of the form \( \mathbb{Z}_{p^n} \times \mathbb{Z}_p \) which completes the proof.

We can state a similar theorem as Theorem 2.4 as follows.

**Theorem 3.2** : Let \( n > 1 \) be an integer and \( p \) be a prime number. Then a group \( G \) is \( \mathbb{Z}_{p^n} \times \mathbb{Z}_p \) if and only if Sub\( (G) \) is an \( n-M_{p+1} \) chain.

One can note that both of the class of all \( n-M_3 \) chains for all integers \( n \) and the class of all \( n-M_{p+1} \) chains for all integers \( n > 1 \) and all odd primes \( p \) are subclasses of the class of all modular lattices which are examples answering to the following open problem.

**Open Problem** : Find a (maximum) subclass \( M \) of modular lattices satisfying these 2 conditions :
(i) \( G \) is a finite abelian group if and only if Sub\( (G) \) is in \( M \), and
(ii) \( L \) is a lattice in \( M \) if and only if \( L \) is isomorphic to Sub\( (G) \) for some finite abelian group \( G \).

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**References**


