On the Study of Nyquist Contour Handling Sampled-Data Control System Real Poles and Zeros

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Abstract
The Nyquist plot is a crucial tool in the analysis and design of linear time-invariant (LTI) and linear shift-invariant (LSI) control systems such as in the relative stability analysis, gain margin, phase margin and robust stability analysis. In the Nyquist plot plane, certain positions must be determined, such as the real axis crossings. These positions are sometimes hidden or ambiguous because of the large span in the magnitude of the Nyquist plot over the entire frequency range. As a result, the Nyquist sketch is introduced as a guide to manually draw the Nyquist plot regarding the qualitative graphical representations such as high frequency asymptote, low frequency asymptote (DC gain point), real and imaginary axis crossings. For LTI control systems or continuous-time control systems, the Nyquist sketch is demonstrated in several studies. However, for LSI control systems or discrete-time control systems, few references mention the Nyquist sketch and only in a vague manner. This study delineates extensively the sketch of the Nyquist plot or the Nyquist sketch for discrete-time control systems extending to the sampled-data control systems when the loop pulse transfer function possesses some real poles and/or zeros outside the unit circle in the z-plane.

Keywords: Nyquist contour; Nyquist sketch; Nyquist plot; relative stability; unit circle; stability of discrete-time systems; Nyquist stability criterion

Nomenclature

\[ GH(z) \]

The loop pulse transfer function.

\[ G_p(s) \]

Transfer function of a continuous-time plant.

\[ H(s) \]

Transfer function of a sensor in the feedback path.

\[ T \]

Sampling period in seconds.

\[ R(z) \]

The z-transform of the input sequence to the system.

\[ C(z) \]

The z-transform of the output sequence of the system.

\[ P(z) = 1 + GH(z) \]

The characteristic equation of the

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1. Introduction

With the publication of “Regeneration Theory” by H. Nyquist [1], who used the term "Regeneration" as synonymous with “feedback”, the general stability problem was found to be solved by the well-known Nyquist stability criterion that he established as well. The Nyquist stability criterion has been used to examine the relative stability of continuous-time control systems [2], [3]. With the graphical tool called “Nyquist plot”, the relative stability analysis of systems involving time delays is obviously comprehensible. Its effectiveness depends upon the correct shape of the Nyquist plot of the loop transfer function. With the introduction of microprocessors and microcontrollers, continuous-time control systems have evolved into the mixture of the continuous-time plants and the discrete-time controllers bridged by the sampling process. The new control systems are called the sampled-data control systems. Researchers facilitated the analysis and design of the sampled-data control systems by converting them into the discrete-time control systems. The analysis and design of discrete-time control systems in frequency domains is done in the z-plane. The relative stability property of the discrete-time control system still relies on the Nyquist stability criterion considering the unit circle in the z-plane as the reference boundary for constructing the Nyquist contour prior to the Nyquist plot as the mapping result. In [4], the Nyquist contour is defined as the unit circle with the imposed constraint of the equal
order of both the numerator and denominator constituting the characteristic equation as a rational function of the complex variable $z$. Then the number of zeros (poles of system) outside the unit circle of the characteristic equation are inferred from the ones inside the unit circle with additional information by using the Principle of Argument theorem. However, there are some situations in which the imposed constraint is invalid, yielding an incorrect result from the relative stability test. In order to overcome the problem, some researchers [5], [6], [7], [8], [9] and [10] redefined the Nyquist contour surrounding the entire area of the $z$-plane excluding the unit circle (enclosing the exterior of the unit circle) without the aforementioned constraint but with some segments traversing on the real axis of the $z$-plane outside the unit circle. These segments make the Nyquist contour exclude real poles and zeroes of $GH(z)$ outside the unit circle leading to the relative stability test error. Although there are software packages that assist in plotting, all the input data must be numerical values and it is very hard to quantitatively explain the infinite quantities. It always happens that the obtained Nyquist plot is unreadable when the system possesses poles on the real axis outside the unit circle in the $z$-plane due to the large span in the magnitude of the polar plot over the entire frequency range. Accordingly, this study delineates extensively the sketch of the Nyquist plot or the Nyquist sketch for discrete-time control systems extending to the sampled-data control systems considering poles and zeroes, especially when some of the real poles and/or zeroes of $GH(z)$ are outside the unit circle in the $z$-plane.

2. Nyquist Contours and Qualitative Graphical Representations

Figure 1. A basic sampled-data control system.

time plant $G_p(s)$ with a zero-order hold, \[ \frac{1-e^{-sT}}{s} \] leading it. $G(s)$ is defined as:

\[ G(s) = \frac{1-e^{-sT}}{s} \cdot G_p(s) \quad (1) \]

The error signal $R(s)-C(s)$ is sampled by a sampler with the sampling period of $T$ s. The feedback path represents the sensor transfer function $H(s)$ . Converting this system to a discrete-time control system results in the pulse transfer function relating the input $R(z)$ to output $C(z)$ as:

\[ \frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)} \quad (2) \]

The characteristic equation $P(z)=1+GH(z)$ describes the behaviors including the relative stability of this system. For the relative stability analysis, the system becomes unstable when there is at least one pole (one zero of $P(z)$) of the system outside the unit circle in the $z$-plane. The Nyquist stability criterion implies the number of zeros of $P(z)$ from the number of encirclements of the Nyquist plot of the loop pulse transfer function $GH(z)$ around point $GH(z)=-1$ in the $GH(z)$ plane. The Nyquist plot can be
Consider the situation when no poles of $GH(z)$ are on the unit circle or on the positive real axis at $z=1$ in the $z$-plane. For illustration, a real pole, $z=p$, and a real zero, $z=z_s$, are given. The Nyquist contour can be drawn as in Figure 2. The contour comprises 4 segments, namely

**Figure 2.** Nyquist contour when no pole of $GH(z)$ on the unit circle or on positive real axis from $z=1$ in the $z$-plane.

### 2.1.1 $C_1$ segment:

This segment is a straight line, a short distance ($\varepsilon$) above the real axis of the $z$-plane, traversing from $z \rightarrow \infty + j\varepsilon$ (point d, e) to $z = 1 + j\varepsilon$ (point a1, c1). It is characterized by

$$z = \frac{1}{r+1} + j\varepsilon \quad \text{..................}(6)$$

Where $-1 \leq r \leq 0$ and $\varepsilon \rightarrow 0 (\varepsilon = 0^+)$. When $r = -1$, $z \rightarrow \infty + j\varepsilon$ and it is represented by point e.
in Figure 2. By the zero-pole-gain form of $GH(z)$,

$$\angle k = \begin{cases} 
0 \text{ rad}; & k \geq 0 \\
\pm \pi \text{ rad}; & k < 0
\end{cases}$$

............... (7)

For most physical systems, $k > 0$ and therefore

When $m < (h + n)$ and $k > 0$,

$$GH(z) = GH(e)$$

$$\approx \frac{z^m}{z^n} e^{j\omega k}$$

$$\approx 0 + j\beta; \beta \rightarrow 0 (\beta = 0^+)$$

............... (8)

When $m > (h + n)$ and $k > 0$,

$$GH(z) = GH(e)$$

$$\approx \frac{z^m}{z^n} e^{j\omega k}$$

$$\rightarrow \infty + j\beta; \beta \rightarrow 0 (\beta = 0^+)$$

............... (9)

When $m = (h + n)$ and $k > 0$,

$$GH(z) = GH(e)$$

$$\approx \frac{z^m}{z^n} e^{j\omega k}$$

$$\rightarrow 1$$

............... (10)

When $r = 0$, $z = 1 + j\epsilon$ or $z \approx 1$ it is represented by point $a_1$. By the polynomial form of $GH(z)$,

$$GH(z) = GH(a_i) \approx \frac{b_m + b_{m-1} + \ldots + b_1 + b_0}{a_n + a_{n-1} + \ldots + a_1 + a_0}$$

............... (11)

On the $C_1$ segment, $\angle(GH(z))$ does not depend on the complex conjugate pole pairs or zero pairs because of the angle cancellation for each pair. Now, considering when $z = z_i^+ + j\epsilon$,

$$|z - z_i| = |z_i^+ + j\epsilon - z_i| = |0^+ + j\epsilon|$$

$$\approx 0$$

............... (12)

$$\angle(z - z_i) = \angle(0^+ + j\epsilon) \approx 0^+ \text{ rad}$$

............... (13)

When $z = z_i + j\epsilon$,

$$|z - z_i| = |z_i + j\epsilon - z_i| = |0 + j\epsilon|$$

$$\approx 0$$

............... (14)

$$\angle(z - z_i) = \angle(0 + j\epsilon) \approx \frac{\pi}{2} \text{ rad}$$

............... (15)

When $z = z_i^- + j\epsilon$,

$$|z - z_i| = |z_i^- + j\epsilon - z_i| = |0^- + j\epsilon|$$

$$\approx 0$$

............... (16)

$$\angle(z - z_i) = \angle(0^- + j\epsilon) \approx -\frac{\pi}{2} \text{ rad}$$

............... (17)

Graphical representation of $\angle(GH(z)) = 0^+ \text{ rad}$ can be depicted as in Figure (3). From Eq. (12), Eq. (13), Eq. (14), Eq. (15), Eq. (16), and Eq. (17), it means that when travelling on $C_1$ segment, $\angle GH(z)$ is changed by an amount of $\frac{\pi}{2} \text{ rad}$ at the position directly over each zero or pole of $GH(z)$ on the real axis of the
z-plane. \( |GH(z)| = 0 \) at the position exactly over each zero, whereas \( |GH(z)| \to \infty \) at the position exactly over each pole. Just after passing the position directly over each zero or pole, \( \angle GH(z) \) is changed by an amount of \( \pi \text{ rad} \) with the same \( |GH(z)| \) as at the position directly over each zero or pole. Consecutively, in the Nyquist sketch, \( |GH(z)| \) is changed abruptly at the position of \( z \) over either the pole or the zero and preserved at that magnitude until \( \angle GH(z) \) is changed by a greater amount of \( \frac{\pi}{2} \text{ rad} \) from that position.

\[
\angle (0^+ + j\varepsilon) = 0^+ 
\]

**Figure 3.** Graphical representation of \( \angle (0^+ + j\varepsilon) = 0^+ \text{ rad} \).

### 2.1.1.2 \( |GH(z)| \) on the intervals over the real poles and zeros of \( GH(z) \)

Moving on the \( C_1 \) segment results in the variation of \( |GH(z)| \) as follows:
- Moving from point \( e \) to a position directly over a zero: \( |GH(z)| \) is varied as \( 0 \to \text{finite} \to 0 \)
- Moving from point \( e \) to a position directly over a pole: \( |GH(z)| \) is varied as \( 0 \to \text{finite} \to \infty \)
- Moving from a position directly over a zero to a position directly over a pole: \( |GH(z)| \) is varied as \( 0 \to \text{finite} \to \infty \)

#### 2.1.2 \( C_{la} \) segment:

This segment is a straight line, a short distance (\( \varepsilon \)) above the real axis of the \( z \)-plane, traversing from \( z = 1 + j\varepsilon \) (point \( a_1 \)) to \( z = 1^- + j\varepsilon, \) \( (z = e^{\jmath 0^\circ}) \) (or point \( a \) on the unit circle) in Figure 2. It is characterized by

\[
z = 1 - r + j\varepsilon \quad \text{…………………(18)}
\]

Where \( 0 \leq r \leq 0^+ \). When \( r = 0, \ z = 1 + j\varepsilon \) (point \( a_1 \)) and no pole or zero of \( GH(z) \) is at \( z = 1, \ GH(a_1) \) is as of Eq.(11). When \( r = 0^+, \ z = 1^- + j\varepsilon \) (point \( a \) ) and no pole or zero of \( GH(z) \) is at \( z = 1, \ GH(a) \) is as of Eq.(11).

### 2.1.3 \( C_2 \) segment:

This segment is characterized by

\[
z = e^{\jmath \Omega} \text{ where } 0 < \Omega \leq (2\pi)^\dagger 
\]
When $\Omega \rightarrow 0(\Omega = 0^\circ)$, $z \rightarrow 1(z = e^{j0^\circ})$ it is represented by point a in Figure 2.

$$GH(z) = GH(a)$$

$$= k \frac{(e^{j0^\circ} - z_1)(e^{j0^\circ} - z_2) \ldots (e^{j0^\circ} - z_m)}{e^{j(0^\circ)}(e^{j0^\circ} - p_1)(e^{j0^\circ} - p_2) \ldots (e^{j0^\circ} - p_n)}$$

$$\approx \frac{b_m + b_{m-1} + \ldots + b_1 + b_0}{a_n + a_{n-1} + \ldots + a_1 + a_0}$$

............... (20)

When $\Omega = \frac{\pi}{2}$, $z = e^{j\frac{\pi}{2}} = j$ it is represented by point q. By the polynomial form of $GH(z)$,

$$GH(z) = GH(q)$$

$$= \frac{b_m j^n + b_{m-1} j^{n-1} + \ldots + b_1 j + b_0}{j^n(a_n j^n + a_{n-1} j^{n-1} + \ldots + a_1 j + a_0)}$$

$$= \text{Re}[GH(q)] + j \text{Im}[GH(q)]$$

............... (21)

$$\angle(GH(q)) = \tan^{-1}\left(\frac{\text{Im}[GH(q)]}{\text{Re}[GH(q)]}\right)$$

............... (22)

$$|GH(q)| = \sqrt{(\text{Re}[GH(q)])^2 + (\text{Im}[GH(q)])^2}$$

............... (23)

When $\Omega = \pi$, $z = -1(z = e^{j\pi})$ it is represented by point b. By the polynomial form of $GH(z)$,

$$GH(z) = GH(b)$$

$$= \frac{(-1)^n b_m + (-1)^{n-1} b_{m-1} + \ldots - b_1 + b_0}{(-1)^n a_n + (-1)^{n-1} a_{n-1} + \ldots - a_1 + a_0}$$

............... (24)

When $\Omega \rightarrow (2\pi)^\circ$, $z \rightarrow 1(z = e^{j(2\pi)^\circ})$ it is represented by point c.

$$GH(z) = GH(c)$$

$$= k \frac{(e^{j(2\pi)^\circ} - z_1)(e^{j(2\pi)^\circ} - z_2) \ldots (e^{j(2\pi)^\circ} - z_m)}{e^{j(2\pi)^\circ}(e^{j(2\pi)^\circ} - p_1)(e^{j(2\pi)^\circ} - p_2) \ldots (e^{j(2\pi)^\circ} - p_n)}$$

$$\cdot \ldots (e^{j(2\pi)^\circ} - z_m)$$

$$\ldots (e^{j(2\pi)^\circ} - p_n)$$

$$\approx \frac{b_m + b_{m-1} + \ldots + b_1 + b_0}{a_n + a_{n-1} + \ldots + a_1 + a_0}$$

............... (25)

From Eq. (20), Eq. (21), Eq. (22), Eq. (23), Eq. (24) and Eq. (25), it is obvious that the $C_2$ segment in the $z$-plane is mapped to a curve in the $GH(z)$ plane starting from point $GH(a)$ on the real axis of the $GH(z)$ plane and moves in the direction (CCW or CW depending upon the number of zeros and poles of $GH(z)$ surrounded by this segment in the $z$-plane) such that the next quadrant in the $GH(z)$ plane is identified by $\angle GH(q)$.

This curve continues to move to point $GH(b)$ on the real axis of the $GH(z)$ plane and ends at point $GH(c)$ which is very close to point $GH(a)$. In the $GH(z)$ plane, it is possible to locate the real axis crossings and imaginary axis crossings from this curve by letting

$$z = x + jy$$

............... (26)

Owing to the $C_2$ segment being the unit circle arc, any point $z$ on $C_2$ must comply with the following relation:

$$x^2 + y^2 = 1$$

............... (27)

By the zero-pole-gain form of $GH(z)$,

$$GH(z) = k \frac{(x + jy - z_1)(x + jy - z_2) \ldots (x + jy - z_m)}{(x + jy)^h(x + jy - p_1)(x + jy - p_2) \ldots (x + jy - p_n)}$$

$$\approx \text{Re}[GH(x + jy)] + j \text{Im}[GH(x + jy)]$$

............... (28)
The real axis crossings are obtained from the coordinates \((x, y)\) satisfying the two following equations:

\[
\text{Im}[GH(x + jy)] = 0 \tag{29}
\]
\[
x^2 + y^2 = 1 \tag{30}
\]

With the calculated \((x, y)\) from Eq. (29) and Eq. (30), the real axis crossings in the \(GH(z)\) plane are expressed as the coordinate \((GH(x + jy), 0)\).

The imaginary axis crossings are obtained from the coordinates \((x, y)\) satisfying the two following equations:

\[
\text{Re}[GH(x + jy)] = 0 \tag{31}
\]
\[
x^2 + y^2 = 1 \tag{32}
\]

With the calculated \((x, y)\) from Eq. (31) and Eq. (32), the imaginary axis crossings in the \(GH(z)\) plane are expressed as the coordinate \((0, GH(x + jy))\).

2.1.4 \(C_{3a}\) segment:
This segment is a straight line, a short distance \((\varepsilon)\) above the real axis of the \(z\)-plane, traversing from \(z = 1^{-} + j\varepsilon\), \((z = e^{j\varepsilon})\) (or point \(c\)) to \(z = 1 + j\varepsilon\) (point \(c_{1}\)) in Figure 2. It is characterized by

\[
z = 1 + r + j\varepsilon \tag{33}
\]

Where \(0^{-} \leq r \leq 0\). When \(r = 0^{-}\), \(z = 1^{-} + j\varepsilon\) (point \(c\)) and no pole or zero of \(GH(z)\) is at \(z = 1\), \(GH(c)\) is as of Eq.(25). When \(r = 0\), \(z = 1 + j\varepsilon\) (point \(c_{1}\)) and no pole or zero of \(GH(z)\) is at \(z = 1\), \(GH(c_{1})\) is as of Eq.(25).

2.1.5 \(C_{3}\) segment:
This segment is characterized by

\[
z = 1 + r + j\varepsilon \quad \text{where} \quad 0 \leq r < \infty \text{ and } \varepsilon \to 0 (\varepsilon = 0^{+}) \tag{34}
\]

When \(r = 0\), \(z \to 1(z = 1 + j\varepsilon)\) and it is represented by point \(c\) in Fig. 2. \(GH(c)\) is as of Eq.(25). When \(r \to \infty\), \(z \to \infty + j\varepsilon\) it is represented by point \(d\). \(GH(d)\) is the same as \(GH(e)\) of Eq. (8), Eq. (9), and Eq. (10), respectively. This means that \(C_{3a}\) segment is mapped by \(GH(z)\) as a point as the one mapped by \(C_{1a}\) segment and \(C_{3}\) segment is mapped by \(GH(z)\) into the \(GH(z)\)-plane with reversed direction from the one mapped by \(C_{1}\) segment.

2.1.6 \(C_{4}\) segment:
This segment is characterized by

\[
z = re^{-j\Omega} \tag{35}
\]

Where \(r \to \infty\) and \(0^{-} \leq \Omega \leq (2\pi)^{-}\). When \(\Omega = 0^{-}\), \(z = re^{-j\Omega^{-}}\) it is represented by point \(d\) in Fig. 2. By the zero-pole-gain form of \(GH(z)\),

When \(m < (h + n)\) and \(k > 0\),

\[
GH(z) = GH(d)
\]
\[
\approx r^{-m} \frac{e^{j\angle(k-(m0)^{-}+(h0)^{-}+(n0)^{-})}}{r^{-h}r^{-n}} \to 0e^{j\angle(k-(m0)^{-}+(h0)^{-}+(n0)^{-})} \\
\to 0e^{j0^{-}}
\]

............... (36)
When \( m > (h + n) \) and \( k > 0 \),
\[
GH(z) = GH(d)
\approx \frac{r^m}{r^h r^n} e^{i \left( k - (m \pi) + (h \pi) + (n \pi) \right)}
\rightarrow \infty e^{i \left( k - (m \pi) + (h \pi) + (n \pi) \right)}
\rightarrow \infty e^{-j(0^\circ)} = \infty e^{j0^\circ}
\]
\[\text{equation (37)}\]

When \( m = (h + n) \) and \( k > 0 \),
\[
GH(z) = GH(d)
\approx \frac{r^m}{r^h r^n} e^{i \left( k - (2m \pi) + (2h \pi) + (2n \pi) \right)}
\rightarrow e^{i \left( k - (2m \pi) + (2h \pi) + (2n \pi) \right)}
\rightarrow e^{j(0^\circ)} = e^{j0^\circ}
\]
\[\text{equation (38)}\]

When \( \Omega = (2\pi)^{-1} \), \( z = re^{j(2\pi)} \) it is represented by point e. By the zero-pole-gain form of \( GH(z) \),

When \( m < (h + n) \) and \( k > 0 \),
\[
GH(z) = GH(e)
\approx \frac{r^m}{r^h r^n} e^{i \left( k - (2m \pi) + (2h \pi) + (2n \pi) \right)}
\rightarrow 0e^{j \left( k - (2m \pi) + (2h \pi) + (2n \pi) \right)}
\rightarrow 0 - j\beta ; \beta \rightarrow 0 (\beta = 0^+) 
\]
\[\text{equation (39)}\]

When \( m > (h + n) \) and \( k > 0 \),
\[
GH(z) = GH(e)
\approx \frac{r^m}{r^h r^n} e^{i \left( k - (2m \pi) + (2h \pi) + (2n \pi) \right)}
\rightarrow \infty e^{i \left( k - (2m \pi) + (2h \pi) + (2n \pi) \right)}
\rightarrow \infty + j\beta ; \beta \rightarrow 0 (\beta = 0^+) 
\]
\[\text{equation (40)}\]
Moreover, let the system have a pole, \( p_i \) and a zero, \( z_j \) located as in Figure 2. The Nyquist plot in this section can be illustrated as in Figure 4.

![Figure 4. Nyquist plot of \( GH(z) \) when no pole of \( GH(z) \) on the unit circle or on positive real axis at \( z = 1 \) in the \( z \)-plane, such a pole \( p_i \) and zero \( z_j \) located as in Figure 2. for \( k > 0 \) and \( m < (h + n) \).](image)

By defining \( C_1 \), \( C_{1a} \), \( C_3 \) and \( C_{3a} \) segments in these manners, only the \( C_2 \) segment plays a major role in the encirclement of \( GH(z) \) around point \( GH(z) = -1 \) in the \( GH(z) \) plane. \( GH(b) \) and \( GH(a) \) or \( GH(c) \) are used to predict the encirclement of \( GH(z) \) around point \( GH(z) = -1 \) in the \( GH(z) \) plane with the direction of rotation indicated by \( GH(q) \).

### 2.2 Nyquist Stability Criterion

After the Nyquist sketch from Section 2.1 is completed, the number of CCW encirclements around point \( GH(z) = -1 \) in the \( GH(z) \) plane is countable as \( N \) (i.e., \(-N\) for CW encirclements). The loop gain \( GH(z) \) has the known \( P \) open loop poles with the unknown \( Z \) zeros of \( P(z) \) inside the Nyquist contour or outside the unit circle in the \( z \)-plane. The rotation direction of the contour in Figure 2, when situated inside the contour, is in CW direction. By the Principle of Argument theorem,

\[
N = P - Z \tag{44}
\]

And therefore,

\[
Z = P - N \tag{45}
\]

Eq. (45) is used to determine the number of zeros of \( P(z) \) inside the contour or outside the unit circle in the \( z \)-plane. The LSI control system (discrete-time control system) is unstable if

\[
Z > 0 \tag{46}
\]

or

\[
N \neq P \tag{47}
\]

Eq. (47) is used to determine the relative stability property of the discrete-time control system.

### 3. Numerical Example

#### 3.1 Example 1.

Considering a sample-data control system in Figure 1. with unity feedback \( H(s) = 1 \), let

\[
G(z) = \frac{1.0}{z - 1.5}
\]

Pole of \( G(z) \) is at : \( z = 1.5 \)

\[
\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{1}{z - 0.5}
\]

The characteristic equation, \( P(z) \) can be expressed as
\[ P(z) = 1 + G(z) = \frac{z - 0.5}{z - 1.5} \]

Zero of \( P(z) \) is at: \( z = 0.5 \)

The Nyquist contour can be demonstrated as in Figure 2. The contour consists of 6 segments, namely

**C₁ segment:** referring to Eq. (6),
\[ z = \frac{1}{r + 1} + j\epsilon \text{ where } -1 \leq r \leq 0 \text{ and } \epsilon \to 0(\epsilon = 0^+) \] . According to Eq. (8)
\[ G(e) \approx 0^+ - j\beta \]

Traversing on this segment to the point directly over the pole at \( z = 1.5 \). This point is expressed as \( z = 1.5 + j\epsilon \)
\[ G(1.5 + j\epsilon) \to \infty \angle \left( -\frac{\pi}{2} \right) \]
Continuing to point \( z = 1.5^- + j\epsilon \)
\[ G(1.5^- + j\epsilon) \to \infty \angle (-\pi) \]

Continuing to point \( z = 1.0 + j\epsilon \)
\[ G(1.0 + j\epsilon) \approx \frac{1}{1 - 1.5} = -2.0 \]

**C₁ₐ segment:** referring to Eq. (18),
\[ z = 1 - r + j\epsilon \text{ where } 0 \leq r \leq 0^+ \text{ and } \epsilon \to 0(\epsilon = 0^+) \] . At point \( z = 1.0 + j\epsilon \)
\[ G(1.0 + j\epsilon) \approx \frac{1}{1 - 1.5} = -2.0 \]

At point \( z = 1.0^- + j\epsilon \)
\[ G(1.0^- + j\epsilon) \approx \frac{1}{1 - 1.5} = -2.0 \]

**C₂ segment:** referring to Eq. (19),
\[ z = e^{j\Omega} \text{ where } 0 < \Omega \leq (2\pi)^+ \] . At point \( z = e^{j(\phi^\circ)} \),
\[ G(e^{j(\phi^\circ)}) \approx \frac{1}{1 - 1.5} = -2.0 \]

At point \( z = e^{j\left(\frac{\pi}{2}\right)} = j \).

\[ G(j) = \frac{1}{j - 1.5} = \frac{-1.5 - j}{\sqrt{1.5^2 + 1}} \]

At point \( z = e^{j(\pi)} = -1 \),
\[ G(-1) = \frac{1}{-1 - 1.5} = -0.4 \]

At point \( z = e^{j(2\pi^+)} \),
\[ G(e^{j(2\pi^+)}) \approx \frac{1}{1 - 1.5} = -2.0 \]

**C₃ₐ segment** is analyzed as \( C₁ₐ \) segment, whereas the mapping by the \( C₃ \) segment is reversed to the one by the \( C₁ \) segment. \( G(d) \) of the \( C₃ \) segment is the same as \( G(e) \) of the \( C₁ \) segment.

**C₄ segment:** referring to Eq. (35),
\[ z = re^{-j\omega} \text{ where } r \to \infty \text{ and } 0^- \leq \Omega \leq (2\pi)^-. \text{ From Eq. (36),} \]
\[ G(d) = G(re^{-j(\phi^\circ)}) \to 0e^{j(\phi^\circ)} \]

From Eq. (39),
\[ G(e) = G(re^{-j(2\pi^+)}) \to 0e^{j(2\pi^+)} \]

\[ \angle G(e) - \angle G(d) = (2\pi) \]

This means that there is an encirclement by \( 2\pi \text{ rad} \) of a miniature circle around the origin of the \( G(z) \) plane starting from point \( G(d) \) and continuing to point \( G(e) \). The Nyquist sketch of this example can be illustrated as in Figure 5.
$P(z)=1+G(z)$
\[
 = \frac{z^4 - 12 z^3 + 56.75 z^2 - 123 z + 92.75}{z^4 - 12 z^3 + 55.75 z^2 - 118.5 z + 87.75}
\]

Zeros of $P(z)$ are at:
\[ z = 1.5252, \ 4.2430, \ 3.1159 \pm 2.1502j \]

The Nyquist contour can be demonstrated as in Figure 2. The contour consists of 6 segments, namely

**C₁ segment:** referring to Eq. (6),
\[ z = \frac{1}{r+1} + j\varepsilon \text{ where } -1 \leq r \leq 0 \text{ and } \varepsilon \rightarrow 0 (\varepsilon = 0^+) \ . \text{ According to Eq. (8)} \]
\[ G(\varepsilon) \approx 0^+ - j\beta \]

Traversing on this segment to the point directly over the pole at $z = 4.5$. This point is expressed as $z = 4.5 + j\varepsilon$
\[ G(4.5 + j\varepsilon) \rightarrow \infty \angle \left( -\frac{\pi}{2} \right) \]

Continuing to point $z = 4.5^+ + j\varepsilon$
\[ G(4.5^+ + j\varepsilon) \rightarrow \infty \angle (-\pi) \]

Continuing to point $z = 2.5 + j\varepsilon$
\[ G(2.5 + j\varepsilon) \rightarrow 0 \angle \left( -\pi + \frac{\pi}{2} \right) = 0 \angle \left( -\frac{\pi}{2} \right) \]

Continuing to point $z = 2.5^- + j\varepsilon$
\[ G(2.5^- + j\varepsilon) \rightarrow 0 \angle \left( -\pi + \frac{\pi}{2} \right) = 0 \angle (0) \]

Continuing to point $z = 2 + j\varepsilon$
\[ G(2 + j\varepsilon) \rightarrow 0 \angle \left( 0 + \frac{\pi}{2} \right) = 0 \angle \left( \frac{\pi}{2} \right) \]

Continuing to point $z = 2^- + j\varepsilon$
\[ G(2^- + j\varepsilon) \rightarrow 0 \angle (\pi) = 0 \angle (\pi) \]

Continuing to point $z = 1.5 + j\varepsilon$
\[ G(1.5 + j\epsilon) \to \infty \angle \left( \frac{\pi - \pi}{2} \right) = \infty \angle \left( \frac{\pi}{2} \right) \]

Continuing to point \( z = 1.5^- + j\epsilon \)
\[ G(1.5^- + j\epsilon) \to \infty \angle \left( \frac{\pi - \pi}{2} \right) = \infty \angle (0) \]

Continuing to point \( z = 1.0 + j\epsilon \)
\[ G(1.0 + j\epsilon) \approx \frac{1^2 - 4.5 + 5}{1^4 - 12(1^3) + 55.75(1^2) - 118.5 + 87.75} \approx 0.1071 \]

\( C_{1a} \) segment: referring to Eq. (18), \( z = 1 - r + j\epsilon \) where \( 0 \leq r \leq 0^+ \) and \( \epsilon \to 0(\epsilon = 0^+) \). At point \( z = 1.0 + j\epsilon \)
\[ G(1.0 + j\epsilon) \approx \frac{1^2 - 4.5 + 5}{1^4 - 12(1^3) + 55.75(1^2) - 118.5 + 87.75} \approx 0.1071 \]
At point \( z = 1.0^- + j\epsilon \)
\[ G(1.0^- + j\epsilon) \approx \frac{1^2 - 4.5 + 5}{1^4 - 12(1^3) + 55.75(1^2) - 118.5 + 87.75} \approx 0.1071 \]

\( C_2 \) segment: referring to Eq. (19), \( z = e^{j\Omega} \) where \( 0 < \Omega \leq (2\pi)^- \). At point \( z = e^{j(\theta^+)} \), \( G(e^{j(\theta^+)} ) \)
\[ \approx \frac{1^2 - 4.5 + 5}{1^4 - 12(1^3) + 55.75(1^2) - 118.5 + 87.75} \approx 0.1071 \]
At point \( z = e^{j\left( \frac{\pi}{2} \right)} = j \),
\[ G(j) = \frac{j^2 - 4.5 j + 5}{j^4 - 12 j^3 + 55.75 j^2 - 118.5 j + 87.75} = \frac{4 - 4.5 j}{1 - 55.75 + 87.75 - 12 j - 118.5 j} = \frac{4 - 4.5 j}{33 - 130.5 j} = 0.0492 + 0.0223 j \]
At point \( z = e^{j(\pi)} = -1 \), \( G(-1) = \frac{(-1)^2 - 4.5 (-1) + 5}{(-1)^4 - 12 (-1)^3 + 55.75 (-1)^2 - 118.5 (-1) + 87.75} = 0.0382 \)
At point \( z = e^{j(2\pi)} \), \( G(e^{j(2\pi)} ) \)
\[ \approx \frac{1^2 - 4.5 + 5}{1^4 - 12(1^3) + 55.75(1^2) - 118.5 + 87.75} \approx 0.1071 \]

\( C_{3a} \) segment is analyzed as \( C_{1a} \) segment, whereas the mapping by the \( C_3 \) segment is reversed to the one by the \( C_1 \) segment. \( G(d) \) of the \( C_3 \) segment is the same as \( G(e) \) of the \( C_1 \) segment.

\( C_4 \) segment: referring to Eq. (35), \( z = re^{j\Omega} \) where \( r \to \infty \) and \( 0^- \leq \Omega \leq (2\pi)^-. \) From Eq. (36),
\[ G(d) = G(re^{-j(\theta^+)}) \to 0e^{j(\theta^+)} \]
From Eq. (39),
\[ G(e) = G(re^{-j(2\pi^-)}) \to 0e^{j(2\pi^-)} \]
\[ \angle G(e) - \angle G(d) = (4\pi) \]
This means that there is an encirclement by \( 4\pi \text{ rad} \) of a miniature circle around the origin of the \( G(z) \) plane starting from point \( G(d) \) and continuing to
point \( G(e) \). The Nyquist sketch of this example can be illustrated as in Figure 6. From Figure 6, \( N = 0 \) and from \( G(z) \), \( P = 4 \), therefore,

\[
Z = P - N = 4 - 0 = 4
\]

As a result, this system is unstable due to the existence of closed-loop poles (zeros of \(1 + G(z) \)) outside the unit circle in the \( z \)-plane complying with the calculation for the zeros of \( P(z) \) at \( z = 1.5252, 4.2430, 3.1159 \pm 2.1502j \).

![Nyquist plot](image)

**Figure 5.** Nyquist plot of \( G(z) = \frac{(z-2)(z-2.5)}{(z-1.5)(z-4.5)(z-3\pm2j)} \)

### 4. Conclusion

On the ground of the Nyquist contours proposed by many literatures, real poles and zeros of the loop pulse transfer function, \( GH(z) \), outside the unit circle in the \( z \)-plane are always excluded, leading to the wrong result of the relative stability test by the Nyquist stability criterion; however, this study proposes a new Nyquist contour that can incorporate those real poles and zeros. In addition, the encirclement around the point \( GH(z) = -1 \) of the mapping by \( GH(z) \) is influenced mainly by the \( C_2 \) segment which is the unit circle in the \( z \)-plane. The relative stability analysis results by this contour are confirmed by two examples in Section 3 that can handle the case of complex conjugate poles and zeros outside the unit circle in the \( z \)-plane as well. Nevertheless, the study scope focuses on the case of real poles and zeros of the loop pulse transfer function, \( GH(z) \), outside the unit circle. Future work will include coverage of the complex conjugate poles and zeros on the unit circle and real poles and/or zeros at \( z = 1 \) in the \( z \)-plane.

### 5. References


