On Classical Quasi-Primary Radical of Submodules and Classical Quasi-Primary Radical Formula of Submodules

Pairote YIARAYONG* and Phakakorn PANPHO

Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand

(*Corresponding author’s e-mail: pairote0027@hotmail.com)

Received: 15 October 2014, Revised: 3 March 2015, Accepted: 5 April 2015

Abstract

In this paper we characterize the classical quasi-primary radical of submodules and classical quasi-primary radical formula of modules over commutative rings with identity. These are extended from radical, radical primary, and radical formula of submodules, respectively. Finally, we obtain necessary and sufficient conditions of a submodule in order to be a top classical quasi-primary radical formula of submodules.

Keywords: Classical quasi-primary submodule, classical quasi-primary radical of submodule, classical quasi-primary radical formula of submodule

Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. We recall that a proper ideal \( P \) of \( R \) is called a primary ideal if \( ab \in P \), where \( a, b \in R \), implies that either \( a \in P \) or \( b^n \in P \), for some positive integer \( n \) [1]. The notion of primary ideal was generalized by Fuchs [1] through defining an ideal \( P \) of a ring \( R \) to be it quasi primary if its radical is a prime ideal, i.e., if \( ab \in P \), where \( a, b \in R \), then either \( a^n \in P \) or \( b^n \in P \), for some positive integer \( n \) (see also [2,3]). A subset \( N \) of the \( R \)-module \( M \) is a submodule, expressed \( N \subseteq M \), if it is a subgroup that is closed under the \( R \)-action: For all \( r \in R \) and \( n \in N \), we have \( rn \in N \). We say that a submodule \( N \) of \( M \) is proper if \( N \neq M \). A proper submodule \( N \) of an \( R \)-module \( M \) is a primary submodule of \( M \) if for \( m \in M \) and \( r \in R \) such that \( rm \in N \), then \( m \in N \) or \( r \in \sqrt{(N : M)} = \{ a \in R \mid a^n M \subseteq N , \text{ for some positive integer } n \} \). An \( R \)-module \( M \) is a primary module if every proper submodule \( N \) of \( M \) is a primary submodule of \( M \) (see, for example, [4-8]). A classical primary submodule in \( M \) as a proper submodule \( N \) of \( M \) such that if \( abK \subseteq N \), where \( a, b \in R \) and \( K \leq M \), then either \( aK \subseteq N \) or \( b^n K \subseteq N \) for some positive integer \( n \). Clearly, in case \( M = R \), where \( R \) is any commutative ring, classical primary submodules coincide with primary ideals [9-11]. A classical quasi-primary submodule in \( M \) as a proper submodule \( N \) of \( M \) such that if \( abK \subseteq N \), where \( a, b \in R \) and \( K \leq M \), then either \( a^p K \subseteq N \) or \( b^q K \subseteq N \) for some positive integer \( n \). The idea of decomposition of submodules into classical primary submodules was introduced by Baziar and Behboodi in [9]. The primary radical of \( N \) in \( M \), denoted by \( \text{prad}_M(N) \), is defined to
be the intersection of all primary submodules containing $N$. Should there be a primary submodule of $M$ containing $N$, then we put $\text{prad}_M(N) = M$. Radicals have been investigated in a number of papers, for example [1,2,12-14]. A classical quasi-primary radical of $N$ in $M$, denoted by $\text{cprad}_M(N)$, is defined to be the intersection of all classical quasi-primary submodules containing $N$. Should there be a classical quasi-primary submodule of $M$ containing $N$, then we put $\text{cprad}_M(N) = M$.

In this note, we shall need the notion of the envelope of a submodule introduced by McCasland and Moore [15]. For a submodule $N$ of an $R$-module $M$, the envelope of $N$ in $M$, denoted by $E_M(N)$, is defined to be the subset $\{rm : r \in R \text{ and } m \in M \text{ such that } r^n m \in N \text{ for some positive integer } n\}$ of $M$ [15-17]. Note that, in general, $E_M(N)$ is not an $R$-module. With the help of envelopes, the notion of the classical primary radical formula is defined as follows: A submodule $N$ of an $R$-module $M$ is said to satisfy the classical quasi-radical formula in $M$, if

$$\langle E_M(N) \rangle = \text{cprad}_M(N).$$

In this paper we introduce the notion of a classical quasi-primary radical of submodules and classical quasi-primary radical formula of modules over a commutative ring with identity. Finally, we obtain necessary and sufficient conditions of a submodule in order to be a top classical quasi-primary radical formula of submodules.

**Basic results**

Let $R = \prod_{i=1}^n R_i$ where each $R_i$ is a commutative ring with identity. Then an ideal $I = \prod_{i=1}^n I_i$ of $P$ is prime if and only if $I_i$ is equal to the corresponding ring $R_i$ and the other is prime. Moreover, any $R$-module $M$ can be uniquely decomposed into a direct product of modules, i.e. $M = \prod_{i=1}^n M_i$, where $M_i = (0,0,0,\ldots,0,1,0,\ldots,0)$ is an $R_i$-module with action;

$$(r_1, r_2,\ldots, r_n)(m_1, m_2,\ldots, m_n) = (r_1 m_1, r_2 m_2,\ldots, r_n m_n),$$

where $r_i \in R_i$ and $m_i \in M_i$ [7].

**Lemma 2.1** [17] Let $N = N_1 \times N_2$ be a submodule of $M$. Then;

$$\langle E_M(N) \rangle = \langle E_{M_1}(N_1) \rangle \times \langle E_{M_2}(N_2) \rangle.$$  (2)

**Proof** See [17].
Corollary 2.2 [17] Let $N = \prod_{i=1}^{n} N_i$ be a submodule of $M$. Then;

$$\langle E_M(N) \rangle = \prod_{i=1}^{n} \langle E_{M_i}(N_i) \rangle.$$ 

(3)

Proof See [17].

Lemma 2.3 [16] If $N$ is a weakly prime submodule, then $\langle E_M(N) \rangle = N$.

(4)

Proof. See [17].

Lemma 2.4 [16] Let $N$ be a semiprime submodule of an $R$-module $M$. Then;

$$\langle E_M(N) \rangle = N.$$ 

(5)

Proof See [16].

Some basic properties of the classical quasi-primary submodules

The results of the following lemmas seem to be at the heart of the theory of classical quasi-primary submodules; these facts will be used so frequently that normally we shall make no reference to this lemma.

Definition 3.1 A proper submodule $N$ of an $R$-module $M$ is said to be a classical quasi-primary submodule of $M$ if $abK \subseteq N$, where $a, b \in R$ and $K \subseteq M$, then either $a^nK \subseteq N$ or $b^nK \subseteq N$ for some positive integer $n$.

Lemma 3.2 Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-module. A submodule $N_1 \times N_2$ is a classical quasi-primary submodule of $M$ if and only if $N_i$ is a classical quasi-primary submodule of $M_i$.

Proof Suppose that $N_1 \times N_2$ is a classical quasi-primary submodule of $R$-module $M$. We will show that $N_1$ is a classical quasi-primary submodule of $M_1$. Clearly, $N_1$ is a proper submodule of $R_i$-module $M_1$. To see the classical quasi-primary property of $N_1$, let $K$ be a submodule of $R_i$-module $M_i$ and $a, b \in R_i$ such that $abK \subseteq N_i$. Then;

$$(a,0)(b,0)(K \times \{0\}) = abK \times \{0\} \subseteq N_1 \times M_2.$$ 

(6)

Since $N_1 \times M_2$ is a classical quasi-primary submodule of $R$-module $M$, it follows that;

$$(a^nK \times \{0\}) = (a,0)^n(K \times \{0\}) \subseteq N_1 \times M_2.$$ 

(7)

or

$$(b^nK \times \{0\}) = (b,0)^n(K \times \{0\}) \subseteq N_1 \times M_2.$$ 

(8)
for some positive integer \( n \), that is \( a^nK \subseteq N_1 \) or \( b^nK \subseteq N_1 \), for some positive integer \( n \). Therefore \( N_1 \) is a classical quasi-primary submodule of \( R_1 \)-module \( M_1 \). Conversely, suppose that \( N_1 \) is a classical quasi-primary submodule of \( R_1 \)-module \( M_1 \). We will show that \( N_1 \times M_2 \) is a classical quasi-primary submodule of \( R \)-module \( M \). Clearly, \( N_1 \times M_2 \) is a proper submodule of \( R \)-module \( M \). To see the classical quasi-primary property of \( N_1 \times M_2 \), let \( K \times L \) be a submodule of \( R \)-module \( M \) and \((a_1, a_2) , (b_1, b_2) \in R \) such that;

\[
a_1 b_1 K \times a_2 b_2 L = (a_1, a_2)(b_1, b_2)(K \times L) \subseteq N_1 \times M_2. \tag{9}
\]

Since \( N_1 \) is a classical quasi-primary submodule of \( R_1 \)-module \( M_1 \) and \( a_1 b_1 K \subseteq N_1 \), we have \( a_i^n K \subseteq N_i \) or \( b_i^n K \subseteq N_i \), for some positive integer \( n \). Therefore;

\[
(a_1, a_2)^n (K \times L) = a_1^n K \times a_2^n L \subseteq N_1 \times M_2 \tag{10}
\]

or

\[
(b_1, b_2)^n (K \times L) = b_1^n K \times b_2^n L \subseteq N_1 \times M_2, \tag{11}
\]

for some positive integer \( n \), and hence \( N_1 \times M_2 \) is a classical quasi-primary submodule of \( R \)-module \( M \).

**Corollary 3.2** Let \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module. A submodule \( M_1 \times N_2 \) is a classical quasi-primary submodule of \( R \)-module \( M \) if and only if \( N_2 \) is a classical quasi-primary submodule of \( R_2 \)-module \( M_2 \).

**Proof** This follows from Lemma 3.2.

**Corollary 3.3** Let \( M = \prod_{i=1}^{n} M_i \), where \( M_i \) is an \( R_i \)-module. A submodule;

\[
M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n \tag{12}
\]

is a classical quasi-primary submodule of \( R \)-module \( M \) if and only if \( N_j \) is a classical quasi-primary submodule of \( R_j \)-module \( M_j \).

**Proof** This follows from Lemma 3.2 and Corollary 3.3.

**Lemma 3.5** Let \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module. If \( N_1 \times \{0\} \) is a classical quasi-primary submodule of \( M_1 \), then \( N_1 \) is a classical quasi-primary submodule of \( M_1 \).
Proof Suppose that \(N_1 \times \{0\}\) is a classical quasi-primary submodule of \(R\)-module \(M\). We will show that \(N_1\) is a classical quasi-primary submodule of \(M_1\). Clearly, \(N_1\) is a proper submodule of \(R_1\)-module \(M_1\). To see the classical quasi-primary property of \(N_1\), let \(K\) be a submodule of \(R_1\)-module \(M_1\) and \(a,b \in R_1\) such that \(abK \subseteq N_1\). Then;

\[
(a,0)(b,0)(K \times \{0\}) = abK \times \{0\} \subseteq N_1 \times \{0\}.
\]  

Since \(N_1 \times M_2\) is a classical quasi-primary submodule of \(R\)-module \(M_1\), it follows that;

\[
(a^nK \times \{0\}) = (a,0)^n(K \times \{0\}) \subseteq N_1 \times \{0\}
\]  

or

\[
(b^nK \times \{0\}) = (b,0)^n(K \times \{0\}) \subseteq N_1 \times \{0\},
\]  

for some positive integer \(n\), that is \(a^nK \subseteq N_1\) or \(b^nK \subseteq N_1\), for some positive integer \(n\). Therefore \(N_1\) is a classical quasi-primary submodule of \(R_1\)-module \(M_1\).

Corollary 3.6 Let \(M = M_1 \times M_2\), where \(M_i\) is an \(R_i\)-module. If \(\{0\} \times N_2\) is a classical quasi-primary submodule of \(R\)-module \(M\), then \(N_2\) is a classical quasi-primary submodule of \(R_2\)-module \(M_2\).

Proof This follows from Lemma 3.5.

Corollary 3.7 Let \(M = \prod_{i=1}^{n} M_i\), where \(M_i\) is an \(R_i\)-module. If \(\{0\} \times \{0\} \times \ldots \times N_j \times \ldots \times \{0\}\) is a classical quasi-primary submodule of \(R\)-module \(M\), then \(N_j\) is a classical quasi-primary submodule of \(R_j\)-module \(M_j\).

Proof This follows from Lemma 3.5 and Corollary 3.6.

Some basic properties of the classical quasi-primary radical formula of submodules

Our starting point is the following lemma:

Lemma 4.1 Let \(M = M_1 \times M_2\), where \(M_i\) is an \(R_i\)-module. If \(W\) is a classical quasi-primary submodule of \(R\)-module \(M\), then \(P = M_1\) or \(P\) is a classical quasi-primary submodule of \(R_i\)-module \(M_1\), where \(P = \{x \in M_1 : (x,0) \in W\}\).

Proof Suppose that \(P \neq M_1\). We will show that \(P\) is a classical quasi-primary submodule of \(R_i\)-module \(M_1\). It is clear that, \(P\) is a proper submodule of \(R_i\)-module \(M_1\). To see the classical quasi-primary property of \(P\), let \(a,b \in R_i\) and \(K\) be submodule of \(M_1\) such that \(abK \subseteq P\). Let \(k \in K\).
Then \( abk \in P \) so that \((a,0)(b,0)(k,0) = (abk,0) \in W \). Thus \((a,0)(b,0)(K \times \{0\}) \subseteq W \). Since \( W \) is a classical quasi-primary submodule of \( M \), we have \((a,0)^n(K \times \{0\}) \subseteq W \) or:

\[(b,0)^n(K \times \{0\}) \subseteq W \]  

for some positive integer \( n \). Thus \((a^n,k,0) = (a,0)^n(k,0) \in W \) or \((b^n,k,0) = (b,0)^n(k,0) \in W \). It follows that \( a^n K \in P \) or \( b^n K \in P \). Therefore \( a^n K \in P \) or \( b^n K \in P \) and hence \( P \) is a classical quasi-primary submodule of \( M \).

**Corollary 4.2** Let \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module. If \( W \) is a classical quasi-primary submodule of \( R \)-module \( M \), then \( P = M_2 \) or \( P \) is a classical quasi-primary submodule of \( R_2 \)-module \( M_2 \), where \( P = \{ x \in M_2 : (0,x) \in W \} \).

**Proof** This follows from Lemma 4.1.

**Corollary 4.3** Let \( M = \prod_{i=1}^{n} M_i \), where \( M_i \) is an \( R_i \)-module and let \( W \) be a classical quasi-primary submodule of \( R \)-module \( M \). If \( P = \{ x \in M_j : (0,0,x,0,0,\ldots,0) \in W \} \), then \( P = M_j \) or \( P \) is a classical quasi-primary submodule of \( R_j \)-module \( M_j \).

**Proof** This follows from Lemma 4.1 and Corollary 4.2.

**Definition 4.4** A classical quasi-primary radical of \( N \) in \( M \), denoted by \( cprad_M(N) \), is defined to be the intersection of all classical quasi-primary submodules containing \( N \). Should there be a classical quasi-primary submodule of \( M \) containing \( N \), then we put \( cprad_M(N) = M \).

**Lemma 4.5** Let \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module and let \( N \) be a submodule of \( R_i \)-module \( M_i \). Then \( m \in cprad_{M_i}(N) \) if and only if \( (m,0) \in cprad_M(N \times \{0\}) \).

**Proof** Suppose that \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module. Let \( N \) be a submodule of \( R_i \)-module \( M_i \) and let \( m \in cprad_{M_i}(N) \). If there is no classical quasi-primary submodule containing \( N \times \{0\} \), then \( cprad_M(N \times \{0\}) = M. \) Therefore \( (m,0) \in M = cprad_M(N \times \{0\}) \). There exists a classical quasi-primary submodule containing \( N \times \{0\} \), then there exists a classical quasi-primary submodule \( W \), with \( N \times \{0\} \subseteq W \). By Lemma 4.1 and \( P = \{ x \in M_1 : (x,0) \in W \} \), we have \( P = M_1 \) or \( P \) is a classical quasi-primary submodule of \( R \)-module \( M_1 \).
Case 1: \( P = M_1 \). Since \( m \in cprad_{M_1}(N) \) we have \( m \in P \). Then \( (m,0) \in W \). Therefore if \( W \) is a classical quasi-primary submodule containing \( N \times \{0\} \), then \( (m,0) \in W \).

Case 2: \( P \neq M_1 \). Since \( P \neq M_1 \), we have \( P \) is a classical quasi-primary submodule of \( R \)-module \( M_1 \). Let \( x \in N \). Then \( (x,0) \in N \times \{0\} \) so that \( x \in P \). It follows that \( N \subseteq P \). We have

\[
cprad_{M_1}(N) \subseteq cprad_{M_1}(P) = P
\]

so that \( m \in P \). Therefore if \( W \) is a classical quasi-primary submodule containing \( N \times \{0\} \), then \( (m,0) \in W \) and hence \( (m,0) \in M = cprad_{M_1}(N \times \{0\}) \).

Corollary 4.6 Let \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module and let \( N \) be a submodule of \( R_2 \)-module \( M_2 \). Then \( m \in cprad_{M_2}(N) \) if and only if \( (0,m) \in cprad_{M_1}(\{0\} \times N) \).

Proof This follows from Lemma 4.5.

Corollary 4.7 Let \( M = \prod_{i=1}^{n} M_i \), where \( M_i \) is an \( R_i \)-module and let \( N \) be a submodule of \( R_j \)-module \( M_j \). Then \( m \in cprad_{M_1}(N) \) if and only if:

\[
(0,0,\ldots,m,0,\ldots,0) \in cprad_{M_1}(\{0\} \times \{0\} \times \ldots \times N \times \{0\} \times \ldots \times \{0\})
\]

Proof This follows from Lemma 4.5 and Corollary 4.6.

Lemma 4.8 Let \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module. If \( N_i \) is a submodule of \( R_i \)-module \( M_i \), then \( cprad_{M_1}(N_i) \times cprad_{M_2}(N_2) \subseteq cprad_{M}(N_i \times N_2) \).

Proof Suppose that \( M = M_1 \times M_2 \), where \( M_i \) is an \( R_i \)-module. Let \( N_i \) be a submodule of \( R_i \)-module \( M_i \). We will show that \( cprad_{M_1}(N_i) \times cprad_{M_2}(N_2) \subseteq cprad_{M}(N_i \times N_2) \). Let:

\[
(x,y) \in cprad_{M_1}(N_1) \times cprad_{M_2}(N_2).
\]

Then \( x \in cprad_{M_1}(N_1) \) and \( y \in cprad_{M_2}(N_1) \). By Lemma 4.4 and Lemma 4.5, we have:

\[
(x,0) \in cprad_{M}(N_1 \times \{0\}) \subseteq cprad_{M}(N_1 \times N_2)
\]

and
Then;

\[(x, y) = (x, 0) + (0, y) \in \text{cprad}_{M}(N_{1} \times N_{2})\]  \hspace{1cm} (22)

and hence \(\text{cprad}_{M_{1}}(N_{1}) \times \text{cprad}_{M_{2}}(N_{2}) \subseteq \text{cprad}_{M}(N_{1} \times N_{2})\).

**Corollary 4.9** Let \(M = \prod_{i=1}^{n} M_{i}\), where \(M_{i}\) is an \(R_{i}\)-module. If \(N_{i}\) be a submodule of \(R_{i}\)-module \(M_{i}\), then \(\prod_{i=1}^{n} \text{cprad}_{M_{i}}(N_{i}) \subseteq \text{cprad}_{M}(\prod_{i=1}^{n} N_{i})\).

**Proof** This follows from Lemma 4.8.

**Theorem 4.10** Let \(M = M_{1} \times M_{2}\), where \(M_{i}\) is an \(R_{i}\)-module. If \(N\) is a submodule of \(R_{1}\)-module \(M_{1}\), then \(\text{cprad}_{M_{1}}(N) \times \text{cprad}_{M_{2}}(M_{2}) = \text{cprad}_{M}(N \times M_{2})\). \hspace{1cm} (23)

**Proof** Suppose that \(M = M_{1} \times M_{2}\), where \(M_{i}\) is an \(R_{i}\)-module. Let \(N\) be a submodule of \(R_{1}\)-module \(M_{1}\). By Lemma 4.8, we have \(\text{cprad}_{M_{1}}(N) \times \text{cprad}_{M_{2}}(M_{2}) \subseteq \text{cprad}_{M}(N \times M_{2})\). We will show that;

\(\text{cprad}_{M_{1}}(N) \times \text{cprad}_{M_{2}}(M_{2}) \supseteq \text{cprad}_{M}(N \times M_{2})\). \hspace{1cm} (24)

If there is no classical quasi-primary submodule containing \(N\), then \(\text{cprad}_{M_{1}}(N) = M_{1}\). Then;

\(\text{cprad}_{M}(N \times M_{2}) \subseteq \text{cprad}_{M_{1}}(N) \times \text{cprad}_{M_{2}}(M_{2})\). \hspace{1cm} (25)

If there are classical quasi-primary submodules containing \(N\), then there exists \(W\) a classical quasi-primary submodule of \(M_{1}\) containing \(N\). Then \(W \times M_{2}\) is a classical quasi-primary submodule of \(M\) containing \(N \times M_{2}\). Let \(P\) be a classical quasi-primary submodule of \(M\) containing \(N \times M_{2}\). Then;

\(N \times M_{2} \subseteq \text{cprad}_{M_{1}}(N_{1}) \times M_{2} = \text{cprad}_{M_{1}}(N) \times \text{cprad}_{M_{2}}(M_{2})\). \hspace{1cm} (26)

Therefore;

\(\text{cprad}_{M}(N \times N_{2}) \subseteq \text{cprad}_{M_{1}}(N) \times \text{cprad}_{M_{2}}(M_{2})\) \hspace{1cm} (27)

and hence \(\text{cprad}_{M_{1}}(N) \times \text{cprad}_{M_{2}}(M_{2}) = \text{cprad}_{M}(N \times M_{2})\).
Corollary 4.11 Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-module. If $N$ is a submodule of $R_2$-module $M_2$, then;

$$cprad_M(M_2 \times N) = cprad_{M_1}(M_2) \times cprad_{M_2}(N).$$  \hspace{1cm} (28)

Proof This follows from Lemma 4.10.

Corollary 4.12 Let $M = \prod_{i=1}^{n} M_i$, where $M_i$ is an $R_i$-module. If $N_i$ be a submodule of $R_i$-module $M_i$, then;

$$\prod_{i=1}^{n} cprad_{M_i}(N_i) = cprad_M(\prod_{i=1}^{n} N_i).$$  \hspace{1cm} (29)

Proof This follows from Lemma 4.10 and Corollary 4.11.

Theorem 4.13 Let $M = M_1 \times M_2$, where $M_i$ is an $R_i$-module. If $N_1$ is a classical quasi-primary submodule of $M_1$, then $N_1$ is to satisfy the classical quasi-primary radical formula in $M_1$ if and only if $N_1 \times M_2$ is to satisfy the classical quasi-primary radical formula in $M$.

Proof Suppose that $N_1$ is a classical quasi-primary submodule of $M_1$ and $N_1$ is to satisfy the classical quasi-primary radical formula in $M_1$. We will show that $N_1 \times M_2$ is to satisfy the classical quasi-primary radical formula in $M$. Since $N_1$ is a classical primary submodule of $M_1$, it follows that;

$$cprad_M(N_1 \times M_2) = cprad_{M_1}(N_1) \times cprad_{M_2}(M_2)$$

$$= \left\langle E_{M_1}(N_1) \right\rangle \times M_2$$

$$= \left\langle E_{M_1}(N_1 \times M_2) \right\rangle.$$  \hspace{1cm} (30)

Therefore $N_1 \times M_2$ to satisfy the classical quasi-primary radical formula in $M$. Conversely, suppose that $N_1$ is a classical quasi-primary submodule of $M_1$ and $N_1 \times M_2$ to satisfy the classical quasi-primary radical formula in $M$. We will show that $N_1$ is to satisfy the classical quasi-primary radical formula in $M_1$. Since $N_1 \times M_2$ is a classical quasi-primary prime submodule of $M$, it follows that;

$$\left\langle E_{M_1}(N_1) \right\rangle \times M_2 = \left\langle E_{M}(N_1 \times M_2) \right\rangle = cprad_{M_1}(N_1) \times cprad_{M_2}(M_2).$$  \hspace{1cm} (31)
Then $e_{prad, M_1}(N_1) = \langle E_{M_1}(N_1) \rangle$ and hence $N_1$ is to satisfy the classical quasi-primary radical formula in $M_1$.

**Theorem 4.14** Let $M = \prod_{i=1}^{n} M_j$, where $M_j$ is an $R_j$-module. If $N_j$ is a classical quasi-primary submodule of $M_j$, then $N_j$ satisfies the classical quasi-primary radical formula in $M_j$ if and only if $M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$ to satisfy the classical quasi-primary radical formula in $M$.

**Proof** This follows from Theorem 4.13.

**Conclusions**

Many new classes of classical quasi-primary radical formula of modules over commutative rings with identity have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. This article investigates the classical quasi-primary submodule, classical quasi-primary radical of submodule, classical quasi-primary radical formula of submodule of modules over commutative rings with identity. Some characterizations of classical quasi-primary radical of submodule and classical quasi-primary radical formula of submodule are obtained. Finally, we obtain necessary and sufficient conditions of a submodule in order to be a classical quasi-primary radical formula of submodules.

**Acknowledgements**

The authors gratefully acknowledge helpful suggestions of the referees.

**References**
