An Iteration Approach for the Scattering of Non-Singular and Singular Potentials
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ABSTRACT
A systematic iterative approach is presented to evaluate the scattering lengths for a few non-singular and singular potentials of finite range. In particular, square-well, Coulomb and Centrifugal potentials have been considered for the case of non-singular potentials and for the singular potentials an inverse fourth power and two logarithmic singular potentials have been studied.

Keywords: iteration, scattering length, non-singular and singular potentials.

1. INTRODUCTION
It is customary in the scattering theory to refer to the zero energy scattering amplitude as the scattering length [1]. In recent years, the variable phase approach formulated by Calogero [2] has found wide applications in determining the phase shift, scattering matrix and scattering amplitude in the non-relativistic quantum theory. Flügge [3] has utilized Calogero’s technique for calculating the scattering lengths of square-well and Yukawa potentials.

Since the solutions of the radial part of the Schrödinger equation for a singular potential \( gV(r) \) are not analytic functions of \( g \) at \( g=0 \), thus in both the attractive and repulsive singular potentials, mathematical difficulties are inherent in regard to the techniques of their solutions. Hence the usual methods of obtaining scattering amplitudes - the Born approximation - fails, and some other calculation method had to be devised.

An important paper by Feinberg and Pais [4] helped to stimulate interest in singular potential theory. In this work the authors have argued that in as much as real world interactions were likely to be highly singular rather than regular (non-singular) potentials might be more relevant physically. Further, in applications involving elementary particle scattering, repulsive singular potentials have been used to simulate the strong repulsion which characterizes the short-range part of interaction between such particles.

Recent investigations on the study of scattering on singular potentials have been used as testing ground to obtain greater insight into the technique of perturbation [5-11]. The perturbation has been applied to the calculations of scattering length of various repulsive singular potentials. However, it is observed that the perturbation technique fails to render the scattering length for logarithmic singular
potentials for which closed solutions cannot also be obtained. It may be noted that the singular nature of the potential manifests itself conspicuously in the coupling constant dependence of the scattering functions and (s-wave) scattering length. Specifically, the singularity of the potential at \( r=0 \) is reflected in the small \( g \) behaviour of the scattering length. In contrast, the large \( g \) behaviour of the scattering length reflects the ‘tail’ i.e., the large \( r \) behaviour of the potential. One readily recognizes that the strong coupling limit emphasizes the effect of the ‘tail’ of the potential which is first encountered by the incident particle, especially for a potential which is repulsively singular. In the weak coupling limit, on the other hand, the particle is not much affected by the ‘tail’ of the potential and for a repulsive singular potential registers the effect of the strong repulsion near the origin.

Esposito [12] has studied some aspects of scattering from singular potentials in quantum mechanics. They used the technique of Bertocchi et al [13], with emphasis on the polydromy properties of the wave function, applied to an arbitrary number of spatial dimensions, say \( q \), when the potential admits a Laurent series expansion. Killingbeck et al [14] have presented a simple numerical method for the singular potentials. They have shown that one-parameter family of coordinate transformation leads to a simple finite difference method which gives highly accurate energies for the Schrödinger equation in which potential consists of a smooth term plus a perturbing term which is singular in the origin.

The object of this paper is to introduce a systematic iteration-perturbation approach for the determination of scattering length for certain nonsingular potentials and singular potentials of finite range. The formulation of the method has been discussed in section 2.

The procedure developed in section 2 is then applied to the cases of non singular potentials. This has been done in section 3. In section 4, the standard scattering length equation is solved by the iteration method for an inverse fourth power and two logarithmic singular potentials by applying a suitably chosen cut-off in accordance with the facts mentioned above.

2. FORMULATION OF THE METHOD

The scattering parameter characterizing zero energy s-wave scattering is the scattering length ‘\( a' \)’. It may be obtained from the first order differential equation [15,16]

\[
a'(r) = -V(r)[r + a(r)]^2
\]  

(2.1)

with boundary condition \( a(0) = 0 \).

We make use of the iteration perturbation method [17] which involves the substitution of the unperturbed function for the unknown function under the integral for obtaining the first order perturbation. The second order correction to the first order approximation is then obtained by substituting this first approximation for the unknown function, and so on.

Thus for obtaining the expression for the scattering length, we write equation (2.1) as

\[
a'(r) = -V(r)[r^2 + 2ra(r) + a^2(r)]
\]  

(2.2)

Now \( a'(r) \) in the first order approximation can be written as

\[
a'(r) = -V(r)r^2,
\]  

(2.3)

which on integration yields the following for the first order ‘interpolating’ scattering length

\[
a_1(r) = -\int_0^r V(r)r^2
dr
\]  

(2.4)

The substitution of \( a_1(r) \) in the second term of (2.2) gives the following second order ‘interpolating’ scattering length

\[
a_2(r) = -2\int_0^r V(r)a_1(r)dr
\]  

(2.5)
Similarly \( a_{III}(r) \) can be obtained by the following relation

\[ a_{III}(r) = -V(r)[a_I(r) + a_{II}(r)] dr \]  

(2.6)

Thus the expression for the scattering length can be finally written as

\[ a_{col}(r) = -a_{I}(r) - a_{II}(r) - a_{III}(r), \]  

(2.7)

where for obtaining \( a, a_I(r), a_{II}(r) \) and \( a_{III}(r) \) have been integrated between the limits \( r = 0 \) and \( r = R \).

For a cut-off potential which vanishes identically for \( r > R \), we simply have

\[ a = a(R) \]  

(2.8)

3. SCATTERING LENGTH FOR NON-SINGULAR POTENTIALS

The square-well, Coulomb and centrifugal potentials are now considered for calculating their scattering lengths. It may, however, be noted that for the centrifugal potential, we have used the improved Born approximation as the unperturbed function for the iteration-perturbation procedure. This has been done due to the singular nature (though this potential is less singular than the inverse fourth power potential) of the centrifugal potential for which the ordinary Born approximation completely breaks down even for small values of the coupling constant \( g \). The forms of the potentials \( V(r) \) and their corresponding expressions for the scattering lengths \( a \), calculated from the iteration-perturbation procedure are given below:

(a) Square-well potential

The square-well potential \( V(r) \) can be written as

\[ V(r) = 2I\theta (R - r) \]  

(3.1)

where, \( I = \frac{V_0}{2} \) and \( r \) is the radius of the well.

Results of calculations for different order contributions to scattering length yield

\[ a_I(r) = -\frac{2I r^3}{3} \]  

(3.2)

\[ a_{II}(r) = +\frac{8I^2 r^5}{15} \]  

(3.3)

and

\[ a_{III}(r) = -\frac{8I^3 r^7}{63} + \frac{64I^4 r^9}{405} - \frac{128I^5 r^{11}}{2475} \]  

(3.4)

Substitution of \( a_I(r), a_{II}(r) \) and \( a_{III}(r) \) in (2.7) yields finally the following for ‘\( a_{SW} \)', the scattering length for the square-well potential where the scale constant \( R \) has been set equal to unity without loss of generality

\[ a_{SW} = -\frac{2I}{3} + \frac{8I^2}{15} - \frac{64I^4}{405} - \frac{128I^5}{2475} \]  

(3.5)

The forms of the potential \( V(r) \) for the Coulomb and centrifugal potentials and their corresponding expressions for the scattering length ‘\( a \)', calculated by the iteration-perturbation procedure, are shown below

(b) Coulomb potential

\[ V(r) = \frac{g}{r} \theta (R - r) \]  

(3.6)

\[ a_{Coul} = -\frac{g}{2} + \frac{g^2}{3} - \frac{g^3}{16} + \frac{g^4}{15} - \frac{g^5}{54}. \]  

(3.7)

(c) Centrifugal potential

\[ V(r) = \frac{g}{r^2} \theta (R - r) \]  

(3.8)

\[ a_{centrifugal} = -g + \frac{2g^2}{1 + 2g} \left( \frac{g^3}{1 + 2g} \right). \]  

(3.9)

The results obtained for the square well potential (Table 1) and Coulomb potential (Table 2) show an improvement over the Born approximation and those for the centrifugal potential (Table 3) over the improved Born approximation. The accuracy of the method, of course, depends upon the selection of suitable unperturbed function. It is interesting to note that the applicability of the Born approximation as an unperturbed function is valid only for small values of \( g \) i.e., for \( |g| \leq 1 \).
4. SCATTERING LENGTH FOR SINGULAR POTENTIALS

We first consider the following inverse fourth power singular potential

\[ V(r) = \frac{g}{(rR)^4} \quad \text{for} \quad r < R \]  

(4.1)

and

\[ V(r) = 0 \quad \text{for} \quad r < R \]  

(4.2)

Changing both the dependent and independent variables for convenience, by setting

\[ a(r) = -RA(x), \quad x = \frac{r}{R}, \]

one gets

\[ A'(x) = -\frac{g}{R^3x} [x + A(x)]^2, \]  

(4.3)

with

\[ A(0) = 0 \]  

(4.4)

and

\[ a = -RA(1) \]  

(4.5)

Calculations of different order contributions to the scattering length yield

\[ A_1(x) = \frac{g}{R^3x} \]  

(4.6)

\[ A_2(x) = -\frac{2g^2}{3R^3x} \]  

(4.7)

and

\[ A_3(x) = \frac{g^3}{5R^3x^3} - \frac{4g^4}{21R^7x^5} + \frac{4g^5}{81R^9x^7} \]  

(4.8)

Final expression of \( a \) for the potential (4.1) with the help of (2.7) and (4.3) to (4.8) can be written as

\[ a = -\frac{g}{R} + \frac{2g^2}{3R^3} - \frac{1g^3}{5R^5} + \frac{4g^4}{21R^7} - \frac{4g^5}{81R^9} \]  

(4.9)

In order to check the validity of equation (4.9), the exact expression for the scattering length of potential (4.1) is calculated and compared with the value calculated in (4.9). The exact s-wave zero energy solution for this potential is given as Wu [7]

\[ \psi(r) = re^{\frac{1}{g^2} - \frac{1}{r}} \quad \text{for} \quad r < R \]  

and

\[ \psi(r) = \alpha r + \beta \quad \text{for} \quad r < R. \]  

(4.10)

Satisfying the following boundary conditions for the scattering

(i) \( \psi(r) \to 0 \) as \( r \to 0 \)

(ii) \( \psi(r) \to r \) as \( r \to \infty \).

The continuity conditions for \( \psi(r) \) and \( \frac{d\psi(r)}{dr} \) at \( r = R \) give

\[ \alpha = e^{-\frac{1}{g^2}} \left[ 1 + \frac{1}{g^2} \right] \]  

and

\[ \beta = -\frac{1}{g^2} e^{-\frac{1}{g^2}} \]  

(4.11)

Equation (4.9) and (4.11) yield finally the following for the exact value of the scattering length

\[ a = -\frac{g^2}{1 + \frac{1}{g^2}}. \]  

(4.12)

Now reinterpreting the cut-off \( R = \frac{1}{g^2} \), yields the following for the scattering length from its exact expression (4.12)

\[ a = 0.5 \frac{1}{g^2}. \]  

(4.13)

Similar reinterpretation of the cut-off in equation (4.9) gives the following for the value of \( 'a' \) obtained by the iteration-perturbation as discussed in this paper

\[ a = 0.4 \frac{1}{g^2}. \]  

(4.14)
There is obviously a close agreement in the value of $a$ obtained in (4.14) by our iteration scheme with that obtained by reinterpreting the cut-off (see Wu [7]).

In order to see whether the iteration scheme discussed in section (2.2) works for other singular potentials also, the following logarithmic singular potentials with pure branch-point type singularities are considered:

$$V(r) = -\frac{g \ln \left(\frac{R}{r}\right)}{r^2}, \quad g < 0 \quad (4.15)$$

and

$$V(r) = -g \frac{\ln \left(\frac{R}{r}\right)}{r^4}, \quad g < 0 \quad (4.16)$$

where these potentials satisfy the condition

$$V(r) \equiv 0 \quad \text{for} \quad r > R.$$  

The potentials (4.15) and (4.16) are short-range potentials and thus will provide well defined scattering length in terms of the cut-off.

It may be noted that potential (4.15) has been discussed by Bertocchi et al [13] to obtain meaningful finite results in non-renormalizable field theories as it provides a model for field theoretic interactions. Arbuzov et al [18] have shown that the wave function for this problem contains an essential singularity at $g = 0$. Further, Calogero and Cassendro [19] have proved that the essential singularity of the wavefunction is also present in the scattering parameter. Brander [20] has proved the existence of Mandelstam representation for this potential. Potential (4.16) has been studied by Wu [7] for understanding the divergent series containing an infinite number of logarithmic factors which is encountered in connection with the higher order leptonic weak interactions. In particular, they calculated an approximate expression for the scattering length valid in the small $g$ limit, using the Born series method.

Following the method adopted for the inverse fourth power potential, the expressions for the scattering lengths for potentials (4.15) and (4.16) can be written as:

$$a = gR \left[1 - 6g + 11g^2 + 0(g^3) \right] \quad (4.17)$$

and

$$a = -\frac{g}{R} + 0.4444 \frac{g^2}{R^2} - 0.0464 \frac{g^3}{R^3} + 0(g^4). \quad (4.18)$$

Assuming $g \ll 1$ (see Calogero and Cassendro [19]), one finds that the result (4.17) is in exact agreement up to $0(g^3)$ with that obtained by using the exact solution for this potential.

In the Appendix, the approximate value of the scattering length has also been obtained, using the integral obtained by Wu [7] by Born series method for the potential (4.16). For small values of $g$, it is observed again that the two approximate results agree remarkably well despite the fact that both have been obtained by two different techniques. This shows the correctness of the iteration scheme presented in this paper.

To obtain the exact value of the scattering length Wu [7] suggested that by suitably reinterpreting the cut-off, one arrives essentially at a correct result. Following this view point, the cut off is set as

$$R = \frac{1}{g^2 \theta^\frac{1}{2}} \quad (4.19)$$

where $\theta$ is defined as $\theta = (\ln \tau)^4$ and $\tau$ is defined by equation $g \tau^4 \ln \tau = 1$. Equation (4.18) with the help of (4.19) leads to the following expression for the scattering length

$$a = -\frac{g}{R} -\frac{1}{2} \left[ 1 - 0.4444 \theta + 0(\theta^2) \right]. \quad (4.20)$$

If this expression for $a$ is compared with the exact value of the scattering length (see
equation (5.8) of Wu [7], one finds that the two results agree closely, while the scattering length obtained by Wu [7] for this potential by Born series method is not so close to the exact result as obtained here.

5. CONCLUSIONS AND DISCUSSIONS

In this paper an iterative approach to solve the first order differential equation of scattering length has been discussed. The method has been applied successfully to three non-singular potentials and to inverse fourth power and logarithmic, singular potentials.

The results of convergence of the iterative scheme depends on the value of \( g > 0 \) for which the scattering problem can be defined. It can be observed that for \( g < 1 \), various contributions to the value of \( a \) have the following behaviour

\[
|a_I| > |a_{II}| > |a_{III}|
\] (5.1)

Thus, the successive contributions decrease in order of magnitude, thereby making the series convergent. It can be seen that the coefficients of various terms in the series for the scattering length \( a \) decrease in order of magnitude [see for example, equation (4.9) and (4.18) for inverse fourth power and \( \ln r / r^4 \) potentials respectively]. The validity of the present iteration scheme and hence its accuracy depends upon the unperturbed function \( a_I \), which incidentally is the Born approximation (see for example, equation (4.20) for the \( \ln r / r^4 \) potential). It is well-known that the Born approximation gives good results in the domain of small values of coupling constant \( g \).

It may be noted that the main purpose of including the second and the third order iteration terms viz., \( a_{II} \) and \( a_{III} \) in the series for the scattering length \( a \), is to increase its accuracy and the domain of validity. Further, the inclusion of these additional terms in the series will also provide accurate results even for larger values of coupling constant \( g \).

APPENDIX

In this appendix, the approximate scattering length for the potential (4.16) has been computed. Following Wu [7], one gets

\[
a = -g^{\frac{1}{2}} \theta^{\frac{3}{2}} \int \frac{dy}{y^{1/2}} \left[ \left( 1 + \theta \ln y \right) \cosh \left( 1 + \theta \ln y \right)^{1/2} y \right]^2,
\] (A.I)

where the constants \( g, \theta \), and \( \tau \) have been set such that

\[
\theta = (\ln \tau)^{-1} \quad \text{and} \quad g \tau^{-2} \ln \tau = 1.
\] (A.II)

It can be observed that equation (A.II) gives

\[
g = \theta e^{\frac{2}{\theta}}
\] (A.III)

Expanding the integral (A.I) in terms of \( \theta \) yields

\[
a = g^{-\frac{1}{2}} \theta^{-\frac{3}{2}} \left[ 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \right) \right) \right) \right) \right) \right) \right]
\] (A.IV)

Evaluation of the integral yields

\[
a = g^{-\frac{1}{2}} \theta^{-\frac{3}{2}} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \left( 1 - \frac{\theta}{2} \right) \right) \right) \right) \right)
\] (A.V)

For a given value of \( \theta \) the corresponding values of \( g \) can be obtained from (A.III). The values of \( g \) thus obtained provide the check for the values scattering length obtained from equation (4.18) and (A.V).
Table 1. Exact and calculated values of scattering length for the square well potential along with their percentage error.

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Table 2. Exact and calculated values of the scattering length for the Coulomb potential along with their percentage error.

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Table 3. Exact and calculated values of scattering length for the centrifugal potential along with their percentage error.

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