On Graded Prime Submodules
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ABSTRACT
Let $G$ be a monoid with identity $e$, and let $R$ be a $G$-graded commutative ring. Here we study the graded prime submodules of a $G$-graded $R$-module. A number of results concerning these class of submodules are given.

Keywords: graded prime submodules, graded rings.

1. INTRODUCTION
Several authors have extended the notion of prime ideals to modules (see [1] and [2], for example). Let $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module. In this paper we introduce the concepts of graded prime submodules of $M$ and give some of their basic properties. However, the prime and graded prime are different concepts.

Before we state some results, let us introduce some notations and terminologies. Let $G$ be an arbitrary monoid with identity $e$. By a $G$-graded commutative ring we mean a commutative ring $R$ with non-zero identity together with a direct sum decomposition (as an additive group) $R = \bigoplus_{g \in G} R_g$ with the property that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by $G(R)$. Also, we write $h(R) = \bigcup_{g \in G} R_g$. The summands $R_g$ are called homogeneous components and elements of these summands are called homogeneous elements. If $a \in R$, then $a$ can be written uniquely as $\sum_{g \in G} a_g$ where $a_g$ is the component of $a$ in $R_g$. In this case, $R_e$ is a subring of $R$ and $1_g \in R_g$.

Let $R$ be a graded ring and $M$ an $R$-module. We say that $M$ is a $G$-graded $R$-module if there exists a family of subgroups $\{M_g\}_{g \in G}$ of $M$ such that $M = \bigoplus_{g \in G} M_g$ (as abelian groups), and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, here $R_g M_h$ denotes the additive subgroup of $M$ consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g \in G} M_g$. If $M = \bigoplus_{g \in G} M_g$ is a graded module, then, $M$ is an $R$-module for all $g \in G$. Let $M = \bigoplus_{g \in G} M_g$ be a graded $R$-module and $N$ a submodule of $M$. For $g \in G$, let $N_g = \bigcap_{r \in R_g} N_r$. Then $N$ is a graded submodule of $M$ if $N = \bigoplus_{g \in G} N_g$. In this case, $N_g$ is called the $g$-component of $N$ for $g \in G$. Moreover, $M/N$ becomes a $G$-graded module with $g$-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Clearly, $0$ is a graded submodule of $M$. An ideal $I$ of $G(R)$ is said to be graded prime ideal if $h^* R$ and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$.

2. THE RESULTS
Our starting point is the following lemma:

Lemma 2.1. Let $R$ be a $G$-graded ring, and $M, N$ be graded $R$-modules. Then $(N : R M) = \{r \in R : r M \subseteq N\}$ is a graded ideal of $G(R)$.

Proof. Since $\bigoplus_{g \in G} (N : R M)_g \subseteq (N : R M)$ is trivial, we will prove the reverse inclusion. Let $a = \sum_{g \in G} a_g \in (N : R M)$. It is enough to show that
\[ a_i M \subseteq N \] for all \( b \in G \). Without loss of generality we may assume that \( a = \sum_{i \in I} a_i \) where \( a_i \neq 0 \) for all \( i = 1, 2, \ldots, m \) and \( a_i = 0 \) for all \( i \not\in \{ h_1, \ldots, h_n \} \). As \( a \in (N :_g M) \), we obtain \( \sum_{i \in I} a_i M \subseteq N \). It suffices to show that for each \( i \), \( a_i m_i \in N \) for any \( m_i \in M \). Since \( M \) is a graded module, we can assume that \( m = \sum_{i} m_i \), with \( m_i \neq 0 \) for all \( j \). Now we show that \( a_i m_i \in N \) for all \( j \). Since for each \( j \), \( am_i \in N \) and \( N \) is a graded module, we obtain \( a_i m_i \in N \) for all \( j \). Thus \( a_i M \subseteq N \) for all \( i = 1, 2, \ldots, m \), as required. 

**Definition 2.2.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module, \( N \) a graded submodule of \( M \) and \( g \in G \).

(i) We say that \( M \) is a \( g \)-torsion-free \( R \)-module whenever \( a \in R \) and \( m \in M \) with \( am = 0 \) implies that either \( m = 0 \) or \( a = 0 \).

(ii) We say that \( M \) is a graded torsion-free \( R \)-module whenever \( a \in b(R) \) and \( m \in M \) with \( am = 0 \) implies that either \( m = 0 \) or \( a = 0 \).

(iii) We say that \( N \) is a \( g \)-pure submodule of the \( R \)-module \( M \) if for each \( a \in R \), \( aN = \bigcap_{g \in G} aM_g \).

(iv) We say that \( N \) is a graded pure submodule of \( M \) if for each \( a \in b(R) \), \( aN = \bigcap_{g \in G} aM_g \).

(v) We say that \( N \) is a \( g \)-prime submodule of the \( R \)-module of \( M \not= M_G \) and whenever \( a \in R \) and \( m \in M \) with \( am \in N \), then either \( m \in N \) or \( a \in (N :_g M)_G \).

(vi) We say that \( N \) is a graded prime submodule of \( M \) if \( N \not= M \) and whenever \( a \in b(R) \) and \( m \in b(M) \) with \( am \in N \), then either \( m \in N \) or \( a \in (N :_g M)_G \).

**Lemma 2.3.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module and \( N \) a graded submodule of \( M \).

(i) If \( N \) is a graded prime submodule of \( M \), then \( N \) is a \( g \)-prime submodule of \( M \) for every \( g \in G \).

(ii) If \( M \) is a graded torsion-free \( R \)-module, then \( M \) is a \( g \)-torsion-free \( R \)-module for every \( g \in G \).

(iii) If \( N \) is a graded pure submodule of a graded torsion-free \( R \)-module \( M \), then \( N \) is a \( g \)-pure submodule of \( M \) for every \( g \in G \).

**Proof.** (i) Suppose that \( N \) is a graded prime submodule of \( M \). For \( g \in G \), assume that \( am \in N \subseteq N \) where \( a \in R \) and \( m \in M \). Since \( N \) is graded prime it gives either \( m \in N \) or \( a \in (N :_g M) \). If \( m \in N \), then \( m \in N \). If \( a \in (N :_g M) \), then \( aM \subseteq \bigcap_{g \in G} aM_g \). Hence \( a \in (N :_g M)_G \). So \( N \) is a prime submodule of \( M \).

(ii) This part is obvious.

(iii) Assume that \( N \) is a graded pure submodule of \( M \) and let \( a \in R \) and \( g \in G \).

Since \( aN_g \subseteq N \cap aM_g \) is trivial, we will prove the reverse inclusion. Take any \( am \in N \cap aM_g \) where \( m \in M \). We can assume that \( am \neq 0 \). Then \( am \in N \cap aM = aN \) since \( N \) is a graded pure submodule. Therefore, \( am = at \) for some \( t \in N \). Hence \( m = tm \) since \( M \) is a graded torsion-free by (ii). Thus \( am \in aN_g \) as required.

**Proposition 2.4.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded torsion-free \( R \)-module and \( N \) a proper graded submodule of \( M \). Then \( N \) is a graded pure submodule of \( M \) if and only if, it is graded prime in \( M \) with \( (N :_g M) = 0 \).

**Proof.** Assume that \( N \) is a graded pure submodule of \( M \) and let \( rm \in N \) with \( r \in (N :_g M) \), where \( r \in b(R) \) and \( m \in b(M) \). Then \( rm \in N \cap rM = rN \), so \( rm = rm \) for some homogeneous element \( n \) of \( N \). It follows that \( n = n \) since \( M \) is graded torsion-free. Suppose that \( a \in (N :_g M) \) with \( a \neq 0 \). Without loss of generality assume \( a = \sum_{g} a_g \), where \( a_g \neq 0 \) for all \( i = 1, 2, \ldots, m \) and \( a_g \neq 0 \) for all \( g \in \{ g_1, \ldots, g_m \} \). As \( N \neq M \), there is a homogeneous element \( m \) of \( M \) such that \( m \not\in N \) and \( a_g m \not\in N \) for all \( i = 1, 2, \ldots, m \) since \( N \) is a graded submodule. Since for every \( i \), \( a_g m = a_g N \), there exists a homogeneous element \( b \) of \( N \) such that \( a_g m = a_g b \). Thus \( m = b \in N \) since \( M \) is graded torsion-free, which is a contradiction. So \( (N :_g M) = 0 \).

Conversely, assume that \( N \) is graded prime in \( M \) with \( (N :_g M) = 0 \) and let \( a \in b(R) \). It is enough to show that \( aM \cap N \subseteq aN \). Let \( ax \in aM \cap N \) where \( x \in M \). We can assume
that \(ax \neq 0\). There are non-zero homogeneous elements \(x_0, \ldots, x_n\) of \(M\) such that \(ax_0, \ldots, ax_n \in N\) since \(N\) is a graded submodule. So \(N\) graded prime and \(a \neq 0\) gives \(x_0, \ldots, x_n \in N\). Hence \(x \in N\), which is required.

Let \(R\) be a \(G\)-graded ring and \(M\) a graded \(R\)-module. We say that \(R\) is a graded integral domain whenever \(a, b \in h(R)\) with \(ab = 0\) implies that either \(a = 0\) or \(b = 0\). If \(R\) is a graded ring and \(M\) is a graded \(R\)-module, the subset \(T(M)\) of \(M\) is defined by \(T(M) = \{m \in M : rm = 0\text{ for some }0 \neq r \in h(R)\}\).

Clearly, \(R\) is an integral domain if and only if \(R\) is a graded integral domain, so if \(R\) is a graded integral domain, then \(T(M)\) is a submodule of \(M\).

**Proposition 2.5.** Let \(R\) be a \(G\)-graded ring, \(M\) a graded \(R\)-module and \(P\) a graded ideal of \(G(R)\). Then every graded prime ideal \(P\) of \(M\) is a graded submodule of \(M\).

**Proof.**

(i) If \(R\) is a graded integral domain, then \(T(M) = \{0\}\) is a graded submodule of \(M\).

(ii) If \(R\) is a graded integral domain and \(T(M) \neq M\), then \(T(M)\) is a graded prime submodule of \(M\).

(iii) Let \(R\) be an overring of \(S\) such that \(S\) is a \(G\)-graded ring. Then every graded prime ideal \(P\) of \(R\) is a graded prime submodule of \(S\)-module \(R\) with \((P : _RR) = P \cap S\).

(iv) \(R/P\) is a graded integral domain if and only if \(P\) is a graded prime ideal of \(G(R)\).

**Proof.**

(i) It is enough to show that \(T(M) = \bigoplus_{g \in G}(T(M) \cap M_g)\). Clearly, \(\bigoplus_{g \in G}(T(M) \cap M_g) \subseteq T(M)\). Let \(m = \sum_{g \in G} m_g \in T(M)\). Our goal is to show that \(m \in T(M)\) for all \(g \in G\). Without loss of generality assume \(m = \sum_{i=1}^n m_i\) where \(m_i \neq 0\) for all \(i = 1, \ldots, n\) and \(m_i = 0\) for all \(g \notin \{g_1, \ldots, g_n\}\). Since \(m \in T(M)\), there exists a non-zero element \(r \in h(R)\) such that \(rm_i = 0\), so we get \(r m_i = \ldots = r m_n = 0\). Hence \(m_i \in T(M)\) for all \(i\), as needed.

(ii) Let \(am \in T(M)\) with \(a \notin (T(M) :_RM)\), where \(a \in h(R)\) and \(m \in h(M)\). Then \(a \in R\) and \(m \in M_g\) for some \(g \in G\). Since \(am \in T(M)\), there exists a non-zero element \(b \in h(R)\), say \(b \in R\), such that \(abm = 0\). If \(am = 0\), then \(m \in T(M)\). So suppose that \(am \neq 0\). As \(R\) is a graded integral domain, we get \(0 \neq ab \in R_g \subseteq h(R)\). Hence \(m \in T(M)\). Thus \(T(M)\) is graded prime.

(iii) Let \(ab \in P\) where \(a \in h(S)\) and \(b \in h(R)\). Then either \(a \in P\) or \(b \in P\) since \(P\) is a graded prime ideal of \(G(R)\). If \(a \in P\), then \(a \in (P : _R)\). Otherwise, \(b \in P\). Hence \(P\) is a prime submodule. Finally, the equality \((P : _R) = P \cap S\) is clear.

(iv) The proof is completely straightforward.

**Lemma 2.6.** Let \(R\) be a \(G\)-graded ring, \(M\) a graded \(R\)-module, \(N\) a graded submodule of \(M\) and \(g \in \mathbb{G}\). Then the following assertions are equivalent.

(i) \(N_g\) is a prime submodule of \(M_g\).

(ii) If whenever \(IB \subseteq N_g\) with \(I\) an ideal of \(R_g\) and \(B\) a submodule of \(M_g\) implies that \(I \subseteq (N_g :_{R_g} M_g)\) or \(B \subseteq N_g\).

**Proof.** (i) \(\Rightarrow\) (ii) Suppose that \(N_g\) is a prime submodule of \(M_g\). Let \(IB \subseteq N_g\) with \(x \in B - N_g\). We want to prove that \(I \subseteq (N_g :_{R_g} M_g)\). Let \(a \in I\). Then \(ax \in N_g\), so \(a \in (N_g :_{R_g} M_g)\) since prime.

(ii) \(\Rightarrow\) (i) Suppose that \(c \in N_g\) where \(c \notin R_g\). Take \(I = R_g c\) and \(B = R_g\). Then \(IB \subseteq N_g\), so either \(B \subseteq N_g\) or \(I \subseteq (N_g :_{R_g} M_g)\) by (ii). Hence either \(c \notin N_g\) or \(c \in (N_g :_{R_g} M_g)\). So \(N_g\) is prime.

**Proposition 2.7.** Let \(R\) be a \(G\)-graded ring, \(M\) a graded \(R\)-module, \(N\) a graded prime submodule of \(M_g\) and \(g \in \mathbb{G}\). Then the following hold:

(i) \((N : _gM_g)\) is a prime ideal of \(R_g\).

(ii) \((N : _gM_g)\) is a graded prime ideal of \(G(R_g)\).

**Proof.** (i) By Lemma 2.3, \(N_g\) is a prime submodule of \(M_g\), so \((N_g :_{R_g} M_g)\) \(\neq R_g\). Let \(ab \in (N_g :_{R_g} M_g)\) where \(a, b \in R_g\). Then \(abM_g \subseteq N_g\). If \(bt \in N_g\) for every \(t \in M_g\), then \(b \in (N_g :_{R_g} M_g)\). So suppose that there is an element \(u \in M_g\) such that \(bu \notin N_g\). As \(ab \in N_g\) and \(bn \notin N_g\), we get \(a \in (N_g :_{R_g} M_g)\), as needed.

(ii) As \(N\) is a graded prime submodule of \(M_g\), we get \((N : _gM_g) \neq R_g\). Let \(cd \in (N : _gM_g)\),
where $c, d \in \mathfrak{b}(R)$. Then $cdM \subseteq N$. If $dM \subseteq N$, then $d \in (N : {}^gM)$. So suppose that there exists $m \in M$ such that $dm \notin N$. As $M$ is a graded $R$-module, there is an element $b \in G$ such that $dm \notin N$. Since $cdm \in N$ and $N$ is a graded submodule, we have $cdm \notin N$. Since $N$ is graded prime gives $c \in (N : {}^gM)$ since $dm \notin N$, as required.

**Lemma 2.8.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Assume that $N$ and $K$ are graded submodules of $M$ with $K \subseteq N$. Then $N$ is a graded prime submodule of $M$ if and only if $N/K$ is a graded prime submodule of the $R$-module $M/K$.

**Proof.** Let $N$ be a graded prime submodule of $M$. Then $N/K \neq M/K$. To show that $N/K$ is a prime submodule of $M/K$, let $a(m+K) \in N/K$ where $a \in h(R)$ and $m+K \in h(M/K)$, so $m \in h(M)$ and $am \in N$. Since $N$ is graded prime it gives either $m + K \in (K : {}^gM)$ or $a \in (N : {}^gM) = (N/K : {}^gM)$. Similarly, we can prove that if $N/K$ is graded prime, then $N$ is graded prime. □

**Theorem 2.9.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Assume that $A$ and $B$ are graded submodules of $M$ with $A + B \neq M$. Then $A + B$ is a graded prime submodule of $M$.

**Proof.** Since $(A + B)/B \cong B/(A \cap B)$, we obtain $A + B$ is a graded prime submodule of $M$ by Lemma 2.8. □

**Theorem 2.10.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a graded prime submodule of $M$ with $(N : {}^gM) = P$. Then there is a one-to-one correspondence between graded prime submodules of the $R/P$-module $M/N$ and the graded prime submodules of $M$ containing $N$.

**Proof.** Let $K$ be a graded prime submodule of $M$ containing $N$. Since $K \neq M$ and $P = (N : {}^gM) = (K : {}^gM)$, we get that $K/N$ is a proper $R/P$-submodule of $M/N$. Let $(a+P)(m+N) = am + N \in K/N$ for $a \in h(R)$ and $m \in h(M)$. Then $K$ being graded prime gives either $m \in M$ or $aM \subseteq K$. Hence either $m + N \in K/N$ or $(a+P)(M/N) \subseteq K/N$. Therefore, $K/N$ is a graded prime submodule of $M/N$. Conversely, let $K/N$ be a graded prime submodule of $M/N$. To show that $K$ is a graded prime submodule of $M$, we suppose that $bt \in K$ where $b \in h(R)$ and $t \in h(M)$. Then $(b+P)(t+N) = bt + N \in K/N$. So $K/N$ being graded prime gives either $t \in K$ or $bM \subseteq K$ as required. □

**Theorem 2.11.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module, $N$ a graded submodule of $M$ and $g \in G$. Then the following hold:

(i) $N_g$ is a prime submodule of $M_g$ if and only if $(N : {}_gM_g) = P_g$ is a prime ideal of $R_g$ and $M_g/N_g$ is a $g$-torsion-free $R_g/P_g$-module.

(ii) $N$ is a graded prime submodule of $M$ if and only if $(N : {}^gM) = P$ is a graded prime ideal of $G(R)$ and $M/N$ is a graded torsion-free $R/P$-module.

**Proof.** (i) First suppose that $N_g$ is a prime submodule of $M_g$. Then by Proposition 2.7(i), $P_g$ is a prime ideal of $R_g$ and $M_g/N_g$ is an $R_g/P_g$-module. If $(a+P_g)(m+N) = N_g$, where $a \in R_g$ and $m \in M_g$, then $am \in N$. So either $m \in N_g$ or $a \in P_g$. Hence $m+N_g = N_g$ or $a+P_g = P_g$. Therefore, $M_g/N_g$ is a $g$-torsion-free $R_g/P_g$-module. Conversely, let $P_g$ be a prime ideal of $R_g$ and let $M_g/N_g$ be a $g$-torsion-free $R_g/P_g$-module. Since $P_g = (N : {}_gM_g) = R_g$, $N_g \neq M_g$. To show that $N_g$ is a prime submodule of $M_g$, assume that $ht \in N_g$ for $b \in R_g$, $t \in M_g$. So $(b+P_g)(t+N) = bt + N_g \subseteq N_g$. Hence either $b \in P_g$ or $t \in N_g$ and the proof is complete.

(ii) Let $N$ be a graded prime submodule of $M$. Then by Proposition 2.7(ii), $P$ is a graded prime ideal of $R$ and $M/N$ is an $R/P$-module. Suppose that $p+P \nsubseteq N$ where $p+P \in h(R/P)$ and $p+P \nsubseteq h(M/N)$. Then $p \in N$ for some $p \in h(R)$, $n \in h(M)$. Therefore, $N$ being graded prime gives either $p+P = P$ or $n+N = N$. Hence $M/N$ is graded torsion-free. Conversely, assume that $P$ is a graded prime ideal of $G(R)$ and let $M/N$ be a graded torsion-free $R/P$-module. Clearly, $N \subseteq M$. To see that $N$ is graded prime, assume that $am \in
where \( a \in h(R) \) and \( m \in h(M) \). Then 
\((a+P)(m+N) = N\). So either \( a \in P \) or \( m \in N \). 
Thus \( N \) is graded prime.

**References**


