ORIGINAL ARTICLE

On bi-Γ-ideals in Γ-semigroups

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Abstract

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Songklanakarin J. Sci. Technol., 2007, 29(1) : 231-234

In 1952, R. A. Good and D. R. Hughes introduced the notion of bi-ideals of semigroups and in 1981, the concept of Γ-semigroups was introduced by M. K. Sen. We have known that Γ-semigroups are a generalization of semigroups. In this research, the notion of bi- Γ-ideals in Γ-semigroups is introduced. We show that bi- Γ-ideals in Γ-semigroups are a generalization of bi-ideals in semigroups and we give some properties for bi- Γ-ideals in Γ-semigroups. We give the two definitions as follows : A Γ-semigroup *M* is called a bi-simple Γ-semigroup if *M* is the unique bi-Γ-ideal of *M* and a bi- Γ-ideal *B* of *M* is called a minimal bi- Γ-ideal of *M* if *B* does not properly contain any bi-Γ-ideal of *M*. We show that a bi-Γ-ideal *B* of a Γ-semigroup *M* is a minimal bi- Γ-ideal of *M* if and only if *B* is a bi-simple Γ-semigroup.

Key words : bi- Γ-ideals, Γ-semigroups, bi-simple Γ-semigroups, minimal bi- Γ-ideals

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Recieved, 15 August 2005 Accepted, 1 August 2006

บทคัดย่อ

รณสรรพ์ ชินรัมย์ และ ชุติพร จิโรจน์กุล
บน Γ-อุดมคติไบใน Γ-กึ่งกลุ่ม
ว. สงขลานครินทร์ วทท. 2550 29(1) : 231-234

ในปี 1952 อาร์ เอ กู๊ด และ ดี อาร์ ฮิวส์ ได้นำเสนอแนวคิดเรื่องอุดมคติไบของกึ่งกลุ่มและในปี 1981 แนวความคิดเรื่อง Γ-กึ่งกลุ่มถูกนำเสนอโดยเอ็ม เค เซน เรารู้ว่า Γ-กึ่งกลุ่มเป็นนัยทั่วไปของกึ่งกลุ่ม ในการวิจัยนี้ Γ-อุดมคติไบใน Γ-กึ่งกลุ่มได้รับการแนะนำ เราได้แสดงว่า Γ-อุดมคติไบใน Γ-กึ่งกลุ่มเป็นนัยทั่วไปของอุดมคติไบในกึ่งกลุ่มและเราให้สมบัติบางอย่างของ Γ-อุดมคติไบใน Γ-กึ่งกลุ่ม เราให้บทนิยามสองบทดังต่อไปนี้ เราเรียก Γ-กึ่งกลุ่ม *M* ว่า Γ-กึ่งกลุ่มเชิงเดียวไบ ถ้า *M* เป็น Γ-อุดมคติไบเพียงหนึ่งเดียวเท่านั้นของ *M* และ เราเรียก Γ-อุดมคติไบ *B* ของ *M* ว่า Γ-อุดมคติไบเล็กสุดเฉพาะกลุ่ม ถ้า *B* ไม่บรรจุ Γ-อุดมคติไบ *C* ของ *M* ซึ่ง $B \neq C$ เราแสดงว่า Γ-อุดมคติไบใน Γ-กึ่งกลุ่มเป็น Γ-อุดมคติไบเล็กสุดเฉพาะกลุ่ม ก็ต่อเมื่อ *B* ว่า Γ-กึ่งกลุ่มเชิงเดียวไบ

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Preliminaries

In 1952, R. A. Good and D. R. Hughes have introduced the notion of bi-ideals of semigroups (Good and Hughes, 1952). The first author has studied some properties of bi-ideals in semigroups (Chinram, 2005). Let S be a semigroup. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$.

Example 1.1. Let $S = [0,1]$. Then S is a semigroup under the usual multiplication. Let $B = [0, \frac{1}{2}]$. Then B is a subsemigroup of S. We have that $BSB = [0, \frac{1}{4}] \subseteq B$. Therefore B is a bi-ideal of S.

Example 1.2. Let **N** be the set of all positive integers. Then **N** is a semigroup under the usual multiplication. Let $B = 2\mathbf{N}$. Thus $B\mathbf{N}B = 4\mathbf{N} \subseteq 2\mathbf{N} = B$. Hence B is a bi-ideal of **N**.

In 1981, the concept of Γ-semigroups was introduced by M. K. Sen. Let M and Γ be any two nonempty sets. If there exists a mapping $M \times \Gamma \times M \longrightarrow M$, written the image of (a, γ, b) by $a\gamma b$, M is called a *Γ- semigroup* if M satisfies the identities $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in M$ and $\gamma, \mu \in \Gamma$ (Sen, 1981, Sen and Saha, 1986, Saha, 1987). Let K be a nonempty subset of M. K is called a *sub Γ- semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Example 1.3. Let $M = [0,1]$ and $\Gamma = \{\frac{1}{n} | n$ is a positive integer$\}$. Then M is a Γ-semigroup under the usual multiplication. Next, let $K = [0, \frac{1}{2}]$. We have that K is a nonempty subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub Γ-semigroup of M.

Example 1. 4. Let S be a semigroup and $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a1b = ab$ for all $a, b \in S$. Then S is a Γ-semigroup.

From Example 1.4, we have seen that every semigroup is a Γ-semigroup where $\Gamma = \{1\}$. Then Γ-semigroups are a generalization of semigroups.

In this research, we generalize bi-ideals of semigroups to bi-Γ-ideals in Γ-semigroups.

Main results

Let M be a Γ-semigroup. A sub Γ-semigroup B of M is called a *bi-Γ-ideal* of M if $B\Gamma M\Gamma B \subseteq B$.

Example 2.1. Let S be a semigroup, and $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a1b = ab$ for all $a, b \in S$. From Example 1.4, we have known that S is a Γ-semigroup. Let B be a bi-ideal of a semigroup S. Thus $BSB \subseteq B$. Since $\Gamma = \{1\}$, $B\Gamma S\Gamma B = BSB \subseteq B$. Hence B is a bi- Γ-ideal of S.

Example 2.1 implies that bi- Γ-ideals in Γ-semigroups are a generalization of bi-ideals in semigroups (for a suitable Γ).

Theorem 2.1. Let M be a Γ-semigroup and B_i a bi-Γ-ideal of M for all $i \in I$. If $\bigcap_{\iota\in I} B_i \neq \varnothing$,then $\bigcap_{\iota\in I} B_i$ is a bi-Γ-ideal of M.

Proof. Let M be a Γ-semigroup and B_i a bi-Γ-ideal of M for all $i \in I$. Assume that $\bigcap_{\iota\in I} B_i \neq \varnothing$. Let $a, b \in \bigcap_{\iota\in I} B_i$, $m \in M$ and $\gamma, \mu \in \Gamma$. Then $a, b \in B_i$ for all $i \in I$. Since B_i is a bi-Γ-ideal of M for all $i \in I$, $a\gamma b \in B_i$ and $a\gamma m\mu b \in B_i\Gamma M\Gamma B_i \subseteq B_i$ for all $i \in I$. Therefore $a\gamma b \in \bigcap_{\iota\in I} B_i$ and $a\gamma m\mu b \in \bigcap_{\iota\in I} B_i$. Hence $\bigcap_{\iota\in I} B_i$ is a bi-Γ-ideal of M.

In Theorem 2.1,$\bigcap_{\iota\in I} B_i \neq \varnothing$ is a necessary condition. Let $M = (0, 1)$ and $\Gamma = \{1\}$. Then M is a Γ-semigroup under the usual multiplication. Let **N** be the set of all positive integers. For $n \in \mathbf{N}$, let $B_n = (0, \frac{1}{n})$. It is easy to prove that B_n is a bi-Γ-ideal of M for all $n \in \mathbf{N}$ but $= \bigcap_{n\in N} B_n = \varnothing$.

Let A be a nonempty subset of a Γ-semigroup M. Let $\Im = \{ B \mid B$ is a bi-Γ-ideal of M containing $A\}$. Then $\Im \neq \varnothing$ because $M \in \Im$. Let $(A)_b = \bigcap_{B\in\Im} B$. It is clearly seen that $A \subseteq (A)_b$. By Theorem 2.1, $(A)_b$ is a bi-Γ-ideal of M. Moreover, $(A)_b$ is the smallest bi-Γ-ideal of M containing A. $(A)_b$ is called *the bi-Γ-ideal of M generated by A*.

Theorem 2.2. Let A be a nonempty subset of a Γ-semigroup M.Then

$(A)_b = A \cup A\Gamma A \cup A\Gamma M\Gamma A$.

Proof. Let A be a nonempty subset of a Γ-semigroup M. Let $B = A \cup A\Gamma A \cup A\Gamma M\Gamma A$. Clearly, $A \subseteq B$. We have that $B\Gamma B = (A \cup A\Gamma A \cup A\Gamma M\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma M\Gamma A) \subseteq A\Gamma A \cup A\Gamma M\Gamma A \subseteq B$. Hence B is a sub Γ-semigroup of M.

Since M is a Γ-semigroup, all elements in $B\Gamma M\Gamma B=(A \cup A\Gamma A \cup A\Gamma M\Gamma A)\Gamma M\Gamma(A \cup A\Gamma A \cup A\Gamma M\Gamma A)$ are in the form of $a_1\gamma m\mu a_2$ for some $a_1, a_2 \in A$, $\gamma, \mu \in \Gamma$ and $m \in M$. Thus $B\Gamma M\Gamma B \subseteq A\Gamma M\Gamma A \subseteq B$. Therefore B is a bi-Γ-ideal of M.

Let C be any bi-Γ-ideal of M containing A. Since C is a sub-Γ-semigroup of M and $A \subseteq C$, $A\Gamma A \subseteq C$. Since C is a bi-Γ-ideal of M and $A \subseteq C$, $A\Gamma M\Gamma A \subseteq C$. Therefore $B = A \cup A\Gamma A \cup A\Gamma M\Gamma A \subseteq C$.

Hence B is the smallest bi-Γ-ideal of M containing A. Therefore $(A)_b = B = A \cup A\Gamma A \cup A\Gamma M\Gamma A$, as required.

Example 2.2. Let **N** be the set of all positive integers and $\Gamma = \{5\}$. Then **N** is a Γ-semigroup under usual addition.

(i) Let $A = \{2\}$. We have that $(A)_b = \{2\} \cup \{9\} \cup \{15, 16, 17,...\}$.

(ii) Let $A = \{3, 4\}$. We have that $(A)_b = \{3, 4\} \cup \{11, 12, 13\} \cup \{17, 18, 19,...\}$.

Theorem 2.3. Let M be a Γ-semigroup. Let B be a bi-Γ-ideal of M and A a nonempty subset of M. Then the following statements are true.

(i) $B\Gamma A$ is a bi-Γ-ideal of M.

(ii) $A\Gamma B$ is a bi-Γ-ideal of M.

Proof. (i) We have that $(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A$ and $(B\Gamma A)\Gamma M\Gamma(B\Gamma A) =(B\Gamma A\Gamma M\Gamma B)\Gamma A$. Since B is a bi-Γ-ideal of M, $(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A \subseteq B\Gamma A$ and $(B\Gamma A)\Gamma M\Gamma(B\Gamma A) = (B\Gamma A\Gamma M\Gamma B)\Gamma A \subseteq (B\Gamma M\Gamma B)\Gamma A \subseteq B\Gamma A$.Therefore $B\Gamma A$ is a bi-Γ-ideal of M.

The proof of (ii) is similar to the proof of (i).

Corollary 2.4. Let M be a Γ-semigroup. For a positive integer n, let $B_1, B_2,..., B_n$ be bi-Γ-ideals of M. Then $B_1\Gamma B_2\Gamma...\Gamma B_n$ is a bi-Γ-ideal of M.

Proof. We will prove the corollary by mathematical induction. By Theorem 2.3, $B_1\Gamma B_2$ is a bi Γ-ideal of M. Next, let n be any positive integer such that $k < n$ and assume $B_1\Gamma B_2\Gamma...\Gamma B_k$ is a bi-Γ-ideal of M. We have that $B_1\Gamma B_2\Gamma...\Gamma B_k\Gamma B_{k+1} = (B_1\Gamma B_2\Gamma...\Gamma B_k)\Gamma B_{k+1}$ is a bi-Γ-ideal of M by Theorem 2.3.

Let M be a Γ-semigroup. M is called a *bi-simple Γ-semigroup* if M is the unique bi-Γ-ideal of M. A bi-Γ-ideal B of M is called a *minimal bi-Γ-ideal* of M if B does not properly contain any bi-Γ-ideal of M.

Example 2.3. Let G be a group and $\Gamma = G$. Then $G^n = G$ and $gG = G = Gg$ for all $g \in G$. Then G is a Γ-semigroup under the usual binary operation. It is easy to see that G is the unique bi-Γ-ideal of G. Then G is a bi-simple Γ-semigroup.

Theorem 2.5. Let M be a Γ-semigroup. Then M is a bi-simple Γ-semigroup if and only if $M = m\Gamma M\Gamma m$ for all $m \in M$, where $m\Gamma M\Gamma m$ means $\{m\}\Gamma M\Gamma\{m\}$.

Proof. Let M be a Γ-semigroup.

Assume that M is a bi-simple Γ-semigroup. Let $m \in M$. By Theorem 2.3, $m\Gamma M\Gamma m$ is a bi-Γ-ideal of M. Then $M = m\Gamma M\Gamma m$.

Assume that $M = m\Gamma M\Gamma m$ for all $m \in M$. Let B be a bi-Γ-ideal of M. Let $b \in B$. By assumption, $M = b\Gamma M\Gamma b \subseteq B\Gamma M\Gamma B \subseteq B$. Hence $M = B$. Therefore M is a bi-simple Γ-semigroup.

Theorem 2.6. Let M be a Γ-semigroup and B a bi-Γ-ideal of M. Then B is a minimal bi-Γ-ideal of M if and only if B is a bi-simple Γ-semigroup.

Proof. Let M be a Γ-semigroup and B a bi-Γ-ideal of M.

Assume that B is a minimal bi-Γ-ideal of M. Let C be a bi-Γ-ideal of B. Then $C\Gamma B\Gamma C \subseteq C$. Since B is a bi-Γ-ideal of M, by Theorem 2.3, $C\Gamma B\Gamma C$ is a bi-Γ-ideal of M. Since B is a minimal bi-Γ-ideal of M and $C\Gamma B\Gamma C \subseteq B$, $C\Gamma B\Gamma C = B$. Hence $B = C\Gamma B\Gamma C \subseteq C$, this implies $B = C$. Then B is a bi-simple Γ-semigroup.

Assume that B is a bi-simple Γ-semigroup. Let C be a bi-Γ-ideal of M such that $C \subseteq B$. Then $C\Gamma B\Gamma C \subseteq C\Gamma M\Gamma C \subseteq C$. Therefore C is a bi-Γ-ideal of B. Since B is a bi-simple Γ-semigroup, $C = B$. Hence B is a minimal bi-Γ-ideal of M, as required.

Acknowledgments.

The authors would like to thank the referees for the useful and helpful suggestions.

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