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Original Article

Existence and stability of fractional complex Liu system

Rabha W. Ibrahim*

Institute of Mathematical Sciences, University of Malaya, Kuala Lumpur, 50603 Malaysia.

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Abstract

In this work, we consider the stability and stabilization of complex fractional Liu system. We assume the fractional calculus in sense of the Caputo derivatives (real and complex). The method based on stability theory of fractional-order systems. Numerical solutions are imposed. Moreover, conditions of unique solution are established.

Keywords: fractional calculus, fractional differential equations, fractional complex system, unit disk, analytic solution

1. Introduction

Liu *et al.* (2004) deduced a new chaotic system. Fractional Liu system is studied in (Jun and Chong, 2007), (Wang and Wang, 2007), (Gao, 2011), (Ibrahim, 2013). Complex chaotic systems are discussed by many contributors (Mohmoud, 2012), (Ibrahim & Jalab,2013). The five dimensional complex Lorenz model is often utilized to explain and simulate the physics of tuned lasers. These studies have their origin in the molding expression by Haken (1975) of the isomorphism between the three equations of the real Lorenz model and the three real equations for a single mode laser operating with its resonant cavity tuned to resonance with the material transition. In this work, we study the stability and stabilization of fractional complex Liu model (five equations). The fractional calculus are assumed in sense of the Caputo derivatives. Caputo initial value problem holds for both homogeneous and nonhomogeneous conditions. For this reason choice Caputo derivative is the best fractional derivatives. The method based on stability theory of fractional-order systems. Numerical solutions are computed.

The Riemann-Liouville fractional operators are defined as follows (Podlubny,1999):

* Corresponding author.
Email address: rabhaibrahim@yahoo.com

Definition 1.1 The fractional order integral of the function h of order $\mu > 0$ is defined by

$$I_a^{\mu} h(t) = \int_a^t \frac{(t-\tau)^{\mu-1}}{\Gamma(\mu)} h(\tau) d\tau.$$

When $a = 0$, we write $I_a^{\mu} h(t) = h(t) * \phi_{\mu}(t)$, where (*) denoted the convolution product, $\phi_{\mu}(t) = \frac{t^{\mu-1}}{\Gamma(\mu)}, t > 0$ and $\phi_{\mu}(t) = 0, t \le 0$ and $\phi_{\mu} \to \delta(t)$ as $\mu \to 0$ where $\delta(t)$ is the delta function.

Definition 1.2 The fractional order derivative of the function h of order $0 \le \mu < 1$ is defined by

$$D_a^{\mu} h(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\mu}}{\Gamma(1-\mu)} h(\tau) d\tau = \frac{d}{dt} I_a^{1-\mu} h(t).$$

Definition 1.3 The Caputo fractional derivative of order $\mu > 0$ is defined, for a smooth function $h(t)$

$${}^{c}D^{\mu} h(t) := \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{h^{(n)}(\zeta)}{(t-\zeta)^{\mu-n+1}} d\zeta,$$

where $n = [\mu] + 1$, (the notation $[\mu]$ stands for the largest integer not greater than μ).

In this paper, we study the stability and stabilization of complex fractional Liu system (with five real variables). We assume the fractional calculus in sense of the Caputo derivatives (real and complex). The method based on stability theory of fractional-order systems. Numerical solutions are imposed. In addition, we establish conditions of unique solution. Caputo's derivative admits higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we have compute its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense.

2. Stability of Fractional Complex System

A complex Liu system has been defined in the form

$$\begin{aligned}\dot{\xi_1} &= -\sigma\xi_1 + \sigma\xi_2\\ \dot{\xi_2} &= \rho\xi_1 - \delta\xi_1\xi_3\\ \dot{\xi_3} &= -\gamma\xi_3 + \eta\xi_1\xi_1^*,\end{aligned}$$

where ξ_1 and ξ_2 are complex variables, i.e. $\xi_1 = u_1 + iu_2$, $\xi_1^* = u_1 - iu_2$ and $\xi_2 = u_3 + iu_4, \xi_2^* = u_3 - iu_4$ but $\xi_3 := u_5$ is a real variable and $\rho, \sigma, \delta, \gamma$ and η are real parameters. It assumed that $u_k, k = 1,...,5$ is a function in *t*. Here, we let the following fractional complex Liu system, in sense of the Caputo derivative:

$$\begin{aligned}{}^cD^{\mu_1}\xi_1 &= -\sigma\xi_1 + \sigma\xi_2\\ {}^cD^{\mu_2}\xi_2 &= \rho\xi_1 - \delta\xi_1\xi_3\\ {}^cD^{\mu_3}\xi_3 &= -\gamma\xi_3 + \eta\xi_1\xi_1^*,\end{aligned} \tag{1}$$

where $0 < \mu_j \le 1, j = 1,2,3;$ in this note we let $\mu_j \ge 0.95$. For real variables, system (1) implies

$$\begin{aligned}{}^cD^{\mu_1}u_1 &= -\sigma u_1 + \sigma u_3\\ {}^cD^{\mu_1}u_2 &= -\sigma u_2 + \sigma u_4\\ {}^cD^{\mu_2}u_3 &= \rho u_1 - \delta u_1 u_5\\ {}^cD^{\mu_2}u_4 &= \rho u_2 - \delta u_2 u_5\\ {}^cD^{\mu_3}u_5 &= -\gamma u_5 + \eta(u_1^2 + u_2^2).\end{aligned} \tag{2}$$

In this section, we investigate the stability of the system (2).

Definition 2.1 The zero solution of the equation ${}^cD^{\mu}u = f(t,u(t)), \mu \in (0,1]$ is said to be stable if, for any initial values u_0, there exists $\varepsilon > 0$ such that $\|u(t)\| \le \varepsilon, \forall t > t_0$. The zero is said to be asymptotically stable if it is stable and $\|u(t)\| \to 0, t \to \infty$.

Lemma 2.1 Assume the system of the form

$${}^cD^{\mu}u = Au(t) + h(u(t)), \mu \in (0,1], \tag{3}$$

where $u(t) = (u_1,...,u_n(t))^T \in \mathrm{R}^n$, $A \in \mathrm{R}^{n\times n}$. If $|arg(spec(A))| > \mu\pi/2, \mu\|A\| > 1$, $spec(A)$ denotes the eigenvalues of A and $\|\cdot\|$ denotes the l_2 norm; and $\lim_{u(t)\to 0}(\|h(u(t))\| / \|u(t)\|) = 0$, then the system (3) is asymptotically stable.

Lemma 2.2 Assume the controller system with the linear feedback control input

$${}^cD^{\mu}u = (A + BK)u(t) + h(u(t)), \mu \in (0,1], \tag{4}$$

where $K \in \mathrm{R}^{1\times n}$ is a feedback, $B \in \mathrm{R}^{n\times 1}$, $u(t) = (u_1,...,u_n(t))^T \in \mathrm{R}^n$, $A \in \mathrm{R}^{n\times n}$. If $|arg(spec(A))| > \mu\pi/2$, $\mu\|A + BK\| > 1$, and $\lim_{u(t)\to 0}(\|h(u(t))\| / \|u(t)\|) = 0$, then the system (5) is asymptotically stable.

Theorem 2.1 If $\gamma > 0, \sigma > 0$ then the system (2) is asymptotically stable at the equilibrium point $p_0 = (0,0,0,0,0)$.
Proof. System (2) can be recognized as in the form (3), where

$$A = \begin{pmatrix} -\sigma & 0 & \sigma & 0 & 0\\ 0 & -\sigma & 0 & \sigma & 0\\ \rho & 0 & 0 & 0 & 0\\ 0 & \rho & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\gamma \end{pmatrix} \quad and \quad h(u) = \begin{pmatrix} 0\\ 0\\ -\delta u_1 u_5\\ -\delta u_2 u_5\\ \eta(u_1^2 + u_2^2)\end{pmatrix}.$$

Obviously $h(u)$ satisfies

$$\begin{aligned}\lim_{u(t)\to 0}\frac{\|h(u(t))\|}{\|u(t)\|} &= \lim_{u(t)\to 0}\frac{\sqrt{(-\delta u_1 u_5)^2 + (-\delta u_2 u_5)^2 + \eta^2(u_1^2 + u_2^2)^2}}{\sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2 + (u_4)^2 + (u_5)^2}}\\ &\le \lim_{u(t)\to 0}\frac{\sqrt{\delta^2 u_5^2(u_1^2 + u_2^2) + \eta^2(u_1^2 + u_2^2)^2}}{\sqrt{u_1^2 + u_2^2}}\\ &= \lim_{u(t)\to 0}\sqrt{\delta^2 u_5^2 + \eta^2(u_1^2 + u_2^2)}\\ &= 0.\end{aligned}$$

In addition the characteristic equation of the system satisfies $(\gamma + \lambda)[\lambda^2 + \sigma\lambda - \rho\sigma]^2 = 0$. But $\gamma > 0, \sigma > 0$, therefore

$$|arg(\lambda_i)| > \frac{\pi}{2} > \frac{\pi}{2}\mu, \quad \mu = max(\mu_i, i = 1,2,3). \tag{5}$$

Moreover, for some $\gamma > 0, \sigma > 0$, we have $\|A\|\underline{\mu} > 1$, $\underline{\mu} := min(\mu_i)$. According to Lemma 3.1, it implies that the equilibrium point p_0 of system (2) is asymptotically stable. This ends the proof.

The system (2) can be considered as in (4), we impose the following result:

Theorem 2.2 If $b > 0, \sigma > 0$ then the controlled system (2) is asymptotically stable at the equilibrium point $p_0 = (0,0,0,0,0)$.

Proof. Employing control input $v(t) = BKu(t)$ on (2), where $B = (1,1,1,1,1)^T$ and $K = (1,1,1,1,1)$ such that $|arg(spec(A+BK))| > \frac{\mu\pi}{2}$. Furthermore, $\mu\|A+BK\| > 1$; thus in view of Lemma 3.2, controlled system (2) is asymptotically stable at p_0.

3. Stabilizing p_0

In this section, we fix a controller for fractional-order Liu chaotic system (2) via fractional-order derivative. For this aim, we use the following result which can be found in (Diethelm & Ford, 2002):

Lemma 3.1 The fixed points of the following nonlinear commensurate fractional-order autonomous system:

$$^cD^{\mu}u = f(u), \mu \in (0,1), \tag{6}$$

is asymptotically stable if all eigenvalues (λ) of the Jacobian matrix evaluated at the fixed points satisfy $|arg\lambda| > 0.5\pi\mu$, where $0 < \mu < 1, u \in R^n, f: R^n \to R^n$ are continuous nonlinear vector functions, and the fixed points of this nonlinear commensurate fractional-order system are calculated by solving equation $f(u) = 0$.

Theorem 3.1 Assume the controlled fractional-order Liu chaotic system

$$\begin{aligned} {}^cD^{\mu_1}u_1 &= -\sigma u_1 + \sigma u_3 \\ {}^cD^{\mu_1}u_2 &= -\sigma u_2 + \sigma u_4 \\ {}^cD^{\mu_2}u_3 &= \rho u_1 - \delta u_1 u_5 + f_1(u_1) \\ {}^cD^{\mu_2}u_4 &= \rho u_2 - \delta u_2 u_5 \\ {}^cD^{\mu_3}u_5 &= -\gamma u_5 + \eta(u_1^2 + u_2^2), \end{aligned} \tag{7}$$

where $f_1(u_1) = -k_{11}\,{}^cD^{\mu_1}u_1 - k_{12}u_1$ is the fractional-order controller, and $k_{1i}, i = 1,2$ is the feedback coefficient. If $\eta > 0, \sigma > 0,$

$$1 + k_{11} > 0, \quad and \quad k_{12} = \rho + k_{11}\sigma,$$

then system (7) will asymptotically converge to the unstable equilibrium point p_0.

Proof. The Jacobi matrix of the controlled fractional-order Liu chaotic system (7) at p_0 is

$$J = \begin{pmatrix} -\sigma & 0 & \sigma & 0 & 0 \\ 0 & -\sigma & 0 & \sigma & 0 \\ \rho + k_{11}\sigma - k_{12} & 0 & -k_{11}\sigma & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma \end{pmatrix}.$$

Since $k_{12} = \rho + k_{11}\sigma$, we have

$$J = \begin{pmatrix} -\sigma & 0 & \sigma & 0 & 0 \\ 0 & -\sigma & 0 & \sigma & 0 \\ 0 & 0 & -k_{11}\sigma & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma \end{pmatrix}.$$

Thus the characteristic equation carries the form

$$(\gamma + \lambda)(\sigma + \lambda)[\lambda^3 + \sigma(k_{11} + 1)\lambda^2 + \lambda\sigma(k_{11}\sigma - \sigma) - \rho\sigma^2 k_{11}] = 0.$$

Since $b > 0, \sigma > 0,$ and $1 + k_{11} > 0$ therefore,

$$|arg\lambda_i| > 0.5\pi\mu, \quad i = 1,...,5,$$

where $\mu := \max \mu_j, j = 1,2,3$. Lemma 4.1 implies that the equilibrium point p_0 of system (7) is asymptotically stable, that is, the unstable equilibrium point p_0 in fractional order Liu system (7) can be stabilized via fractional-order derivative. The proof is completed.

In similar manner of Theorem 4.1, we have the following results:

Theorem 3.2 Assume the controlled fractional-order Liu-chaotic system

$$\begin{aligned} {}^cD^{\mu_1}u_1 &= -\sigma u_1 + \sigma u_3 \\ {}^cD^{\mu_1}u_2 &= -\sigma u_2 + \sigma u_4 \\ {}^cD^{\mu_2}u_3 &= \rho u_1 - \delta u_1 u_5 \\ {}^cD^{\mu_2}u_4 &= \rho u_2 - \delta u_2 u_5 + f_2(u_2) \\ {}^cD^{\mu_3}u_5 &= -\gamma u_5 + \eta(u_1^2 + u_2^2), \end{aligned} \tag{8}$$

where $f_2(u_2) = -k_{21}\,{}^cD^{\mu_1}u_2 - k_{22}u_2$ is the fractional-order controller, and $k_{2i}, i = 1,2$ is the feedback coefficient. If $\gamma > 0, \sigma > 0,$

$$1 + k_{21} > 0, \quad and \quad k_{22} = \rho + k_{21}\sigma,$$

then system (8) will asymptotically converge to the unstable equilibrium point p_0.

Theorem 3.3 Assume the controlled fractional-order Liu chaotic system

$$\begin{aligned} {}^cD^{\mu_1}u_1 &= -\sigma u_1 + \sigma u_3 + f_3(u_3) \\ {}^cD^{\mu_1}u_2 &= -\sigma u_2 + \sigma u_4 \\ {}^cD^{\mu_2}u_3 &= \rho u_1 - \delta u_1 u_5 \\ {}^cD^{\mu_2}u_4 &= \rho u_2 - \delta u_2 u_5 \\ {}^cD^{\mu_3}u_5 &= -bu_5 + \eta(u_1^2 + u_2^2), \end{aligned} \tag{9}$$

where $f_3(u_3) = -k_{31}\,{}^cD^{\mu_2}u_3 - k_{32}u_3$ is the fractional-order controller, and $k_{2i}, i = 1,2$ is the feedback coefficient. If $\gamma > 0, \sigma > 0, k_{32} = \sigma + k_{31}$ and $\sigma + \rho k_{31} > 0$ then system (9) will asymptotically converge to the unstable equilibrium point p_0.

4. Numerical Solution

In this section, we debate a numerical solution of fractional differential equations. All the numerical simulation of fractional-order system in this paper is based on Matignon (1996). Put $h = T/N, t_n = nh, n = 1,...,N$ with the initial condition $(u_1(0),...,u_5(0))$ therefore, the system (2) can be depicted as follows

$$u_1(n+1) = u_1(0) + \frac{h^{\mu_1}}{\Gamma(\mu_1+2)}[\sigma(-u_1^p(n+1)+u_3^p(n+1)) + \sum_{j=0}^{n} A_{1,j,n+1} \times \sigma(-u_1(j)+u_3(j))],$$
$$u_2(n+1) = u_2(0) + \frac{h^{\mu_1}}{\Gamma(\mu_1+2)}[\sigma(-u_2^p(n+1)+u_4^p(n+1)) + \sum_{j=0}^{n} A_{2,j,n+1} \times \sigma(-u_2(j)+u_4(j))],$$
$$u_3(n+1) = u_3(0) + \frac{h^{\mu_2}}{\Gamma(\mu_2+2)}[(\rho u_1^p(n+1) - \delta u_1^p(n+1)u_5^p(n+1))$$
$$+\sum_{j=0}^{n} A_{3,j,n+1}(\rho u_1(j) - \delta u_1(j)u_5(j))]$$
$$u_4(n+1) = u_4(0) + \frac{h^{\mu_2}}{\Gamma(\mu_2+2)}[(\rho u_2^p(n+1) - \delta u_2^p(n+1)u_5^p(n+1))$$
$$+\sum_{j=0}^{n} A_{4,j,n+1}(\rho u_2(j) - \delta u_2(j)u_5(j))]$$
$$u_5(n+1) = u_5(0) + \frac{h^{\mu_3}}{\Gamma(\mu_3+2)}[-\gamma u_5^p(n+1) + \eta(u_1^{2p}(n+1)+u_2^{2p}(n+1))$$
$$+\sum_{j=0}^{n} A_{5,j,n+1}(-\gamma u_5(j) + \eta(u_1^2(j)+u_2^2(j)))],$$

where

$$u_1^p(n+1) = u_1(0) + \sum_{j=0}^{n} B_{1,j,n+1} \times \sigma(-u_1(j)+u_3(j))$$
$$u_2^p(n+1) = u_2(0) + \sum_{j=0}^{n} B_{2,j,n+1} \times \sigma(-u_2(j)+u_4(j))$$
$$u_3^p(n+1) = u_3(0) + \sum_{j=0}^{n} B_{3,j,n+1} \times (\rho u_1(j) - \delta u_1(j)u_5(j))$$
$$u_4^p(n+1) = u_4(0) + \sum_{j=0}^{n} B_{4,j,n+1} \times (\rho u_2(j) - \delta u_2(j)u_5(j))$$
$$u_5^p(n+1) = u_5(0) + \sum_{j=0}^{n} B_{5,j,n+1} \times (-\gamma u_5(j) + \eta(u_1^2(j)+u_2^2(j)))$$

and for $k = 1,...,5$

$$A_{k,j,n+1} = \begin{cases} n^{\mu+1} - (n-\mu)(n+1)^{\mu}, & j = 0 \\ (n-j+2)^{\mu+1} + (n-j)^{\mu+1} - 2(n-j+1)^{\mu+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$

$$B_{k,j,n+1} = \frac{h^{\mu}}{\mu}[(n-j+1)^{\mu} - (n-j)^{\mu}], \quad 0 \le j \le n.$$

The error of this approximation can be computed as follows:

$$|u_i(t_n) - u_i(n)| = o(h^{\mu}), \quad p = min(2, 1+\max \mu_{1,2,3}).$$

5. Synchronizing p_0

We put a feedback controller for the fractional-order Liu chaotic system (2) via fractional-order derivative and obtain the system

$$\begin{aligned} {}^cD^{\mu_1} v_1 &= -\sigma v_1 + \sigma v_3 \\ {}^cD^{\mu_1} v_2 &= -\sigma v_2 + \sigma v_4 \\ {}^cD^{\mu_2} v_3 &= \rho v_1 - \delta v_1 v_5 + V \\ {}^cD^{\mu_2} v_4 &= \rho v_2 - \delta v_2 v_5 \\ {}^cD^{\mu_3} v_5 &= -\gamma v_5 + \eta(v_1^2 + v_2^2), \end{aligned} \tag{10}$$

where

$$V = k_1[{}^cD^{\mu_1} v_1 - D^{\mu_1} u_1] + k_2(v_1 - u_1) + \delta v_1 v_5 - \rho u_1$$

is the fractional-order controller, and $k_i, i = 1,2$ is the feedback coefficient. We need the following result which can be located in Matignon (1996):

Lemma 5.1 The following linear commensurate fractional-order autonomous system

$${}^cD^{\mu} u = Au, \mu \in (0,1), \tag{11}$$

is asymptotically stable if and only if $|arg\lambda| > 0.5\pi\mu,$ is satisfied for all eigenvalues (λ) of matrix A. Furthermore, this system is stable if and only if $|arg\lambda| \ge 0.5\pi\mu,$ is satisfied for all eigenvalues (λ) of matrix A and those critical eigenvalues which satisfy $|arg\lambda| = 0.5\pi\mu,$ have geometric multiplicity one, where $0 < \mu < 1, u \in R^n$, and $A \in R^n \times R^n$.

Theorem 5.1 If $\gamma > 0, \sigma > 0, k_2 = -\rho + k_1\sigma$ and $\sigma(k_1 + \rho) < 1$, then the fractional-order Liu chaotic system (2) and the controlled fractional-order Liu chaotic system (11) attained synchronization via fractional-order derivative.

Proof. Define the synchronization error variables as follows:

$$e_i = v_i - u_i, \quad i = 1,...,5.$$

Therefore, we obtain the system

$$\begin{pmatrix} {}^cD^{\mu_1} e_1 \\ {}^cD^{\mu_1} e_2 \\ {}^cD^{\mu_2} e_3 \\ {}^cD^{\mu_2} e_4 \\ {}^cD^{\mu_3} e_5 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}, \tag{12}$$

where

$$A=\begin{pmatrix} -\sigma & 0 & \sigma & 0 & 0 \\ 0 & -\sigma & 0 & \sigma & 0 \\ \rho-k_1\sigma+k_2 & 0 & k_1\sigma & 0 & 0 \\ 0 & \rho-\delta u_5 & 0 & 0 & -\delta v_2 \\ \eta(v_1+u_1) & \eta(v_2+u_2) & 0 & 0 & -\gamma \end{pmatrix}.$$

Since $k_2=-\rho+k_1\sigma$ then for suitable values of u_i and v_i, we have

$$A=\begin{pmatrix} -\sigma & 0 & \sigma & 0 & 0 \\ 0 & -\sigma & 0 & \sigma & 0 \\ 0 & 0 & k_1\sigma & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma \end{pmatrix}.$$

Thus, the characteristic equation is of the form

$$(\gamma+\lambda)(\sigma+\lambda)(k_1\sigma-\lambda)[\lambda^2+\sigma\lambda-\rho\sigma]=0.$$

Since $\sigma>0,\gamma>0,k_1<0$ yields that $|arg\lambda_i(A)|>\frac{\pi}{2}>\frac{\pi}{2}\mu$, $i=1,...,5$. In virtue of Lemma 6.1, we have that the equilibrium point p_0, of error system (12) is asymptotically stable, and hence the fractional-order Liu chaotic systems (2) and (11) achieved synchronization via fractional-order derivative. The proof is completed.

6. Existence and Uniqueness

In this section we establish the existence and uniqueness of system (2). Let $\xi_j, j=1,2,3$ be in the unit disk $U:=\{z\in\mathbb{C}:|z|\le 1\}$ and $t\in J=[0,T]$. In addition, we assume that $u_1,...,u_5$ are continuous in J; we symbolize this set by $C(J)$. Let $\mathsf{S}=\{(u_1,...,u_5)^{\mathsf{T}}:u_1,...,u_5\in C(J)\}$ with the norm

$$\|\xi_1\|_S=\|u_1\|+\|u_2\|=\sup_{t\in J}|u_1(t)|+\sup_{t\in J}|u_2(t)|,$$

$$\|\xi_2\|_S=\|u_3\|+\|u_4\|=\sup_{t\in J}|u_3(t)|+\sup_{t\in J}|u_4(t)|$$

and

$$\|\xi_3\|_S=\|u_5\|=\sup_{t\in J}|u_5(t)|.$$

Theorem 6.1 Let $\rho=\delta$ and $\eta>-\gamma$ in system (2) withe the initial condition $(u_1(0),...,u_5(0))^{\mathsf{T}}$. If

$$\frac{|\sigma|T^{\mu_1}}{\Gamma(\mu_1+1)}+\frac{|\rho|T^{\mu_2}}{\Gamma(\mu_2+1)}+\frac{|\eta|T^{\mu_3}}{\Gamma(\mu_3+1)}<\frac{1}{4},\quad T\in J$$

then (2) has a unique solution in S.

Proof. The system (2) can be observed as a matrix form

$$\begin{aligned}({}^cD^{\mu_1}u_1,{}^cD^{\mu_1}u_2)^{\mathsf{T}}&=(-\sigma u_1+\sigma u_3,-\sigma u_2+\sigma u_4)^{\mathsf{T}}\\ ({}^cD^{\mu_2}u_3,{}^cD^{\mu_2}u_4)^{\mathsf{T}}&=(\rho u_1-\delta u_1u_5,\rho u_2-\delta u_2u_5)^{\mathsf{T}}\\ {}^cD^{\mu_3}u_5&=-\gamma u_5+\eta(u_1^2+u_2^2).\end{aligned}\quad (13)$$

Operating (13) by I^{μ_1}, I^{μ_2} and I^{μ_3} respectively implies

$$\begin{aligned}(u_1,u_2)^{\mathsf{T}}&=(u_1(0)-\sigma I^{\mu_1}u_1+\sigma I^{\mu_1}u_3,u_2(0)-\sigma I^{\mu_1}u_2+\sigma I^{\mu_1}u_4)^{\mathsf{T}}\\ (u_3,u_4)^{\mathsf{T}}&=(u_3(0)+\rho I^{\mu_2}u_1-\delta I^{\mu_2}u_1u_5,u_4(0)+\rho I^{\mu_2}u_2-\delta I^{\mu_2}u_2u_5)^{\mathsf{T}}\\ u_5&=u_5(0)-\gamma I^{\mu_3}u_5+\eta I^{\mu_3}(u_1^2+u_2^2).\end{aligned}\quad (14)$$

Define the operator $\mathsf{O}:\mathsf{S}\to\mathsf{S}$ by

$$\begin{aligned}\mathsf{O}(u_1,...,u_5)^{\mathsf{T}}&=(u_1(0),...,u_5(0))^{\mathsf{T}}+(-\sigma I^{\mu_1}u_1+\sigma I^{\mu_1}u_3,-\sigma I^{\mu_1}u_2+\sigma I^{\mu_1}u_4,\\ &\rho I^{\mu_2}u_1-\delta I^{\mu_2}u_1u_5,\rho I^{\mu_2}u_2-\delta I^{\mu_2}u_2u_5,\\ &-\gamma I^{\mu_3}u_5+\eta I^{\mu_3}(u_1^2+u_2^2))^{\mathsf{T}}\end{aligned}\quad (15)$$

Because $|u_i|<1$, thus a computation yields

$$\begin{aligned}&\|\mathcal{O}(u_1,...,u_5)^{T}-\mathcal{O}(v_1,...,v_5)^{T}\|_S\\ &=\|(-\sigma I^{\mu_1}(u_1-v_1)+\sigma I^{\mu_1}(u_3-v_3),-\sigma I^{\mu_1}(u_2-v_2)+\sigma I^{\mu_1}(u_4-v_4),\\ &\rho I^{\mu_2}(u_1-v_1)-\delta I^{\mu_2}(u_1u_5-v_1v_5),\rho I^{\mu_2}(u_2-v_2)-\delta I^{\mu_2}(u_2u_5-v_2v_5),\\ &-\gamma I^{\mu_3}(u_5-v_5)+\eta I^{\mu_3}[(u_1^2+u_2^2)-(v_1^2+v_2^2)])^{T}\|_S\\ &\le 4\left(\frac{|\sigma|T^{\mu_1}}{\Gamma(\mu_1+1)}+\frac{|\rho|T^{\mu_2}}{\Gamma(\mu_2+1)}+\frac{|\eta|T^{\mu_3}}{\Gamma(\mu_3+1)}\right)\|(u_1,...,u_5)^{T}-(v_1,...,v_5)^{T}\|_S\end{aligned}$$

where $|v_i|<1$. Hence by the contraction fixed point theorem, the problem (2) has a unique solution in S.

7. Conclusion and Discussion

Using fractional-order derivative (in sense of the Caputo derivatives), we may stabilize the unstable equilibrium points of the complex fractional-order Liu chaotic system and comprehend chaos synchronization for the fractional order Liu chaotic system. The fractional system is taken for different values of $\mu_i, i=1,2,3$ where $\mu_1=0.996$, $\mu_2=0.997,\mu_3=0.998$. The parameters are valued as $\sigma=10,\rho=40,\delta=1,\gamma=2.5$ and $\eta=4$. Figure 1 shows the time series for system (2) while Figure 2 shows the waveform of system (2). Figure 3, shows the asymptotically stabile of the system for different values of (σ,γ), where $\sigma>0$ and $\gamma>0$; in (a)- $(\sigma,\gamma)=(1,0.3),\rho=-0.8$ (b)- $(\sigma,\gamma)=(1,0.3),\rho=-0.8$, in (c)- $(\sigma,\gamma)=(0.1,0.3),\rho=0.8$ and (d)- $(\sigma,\gamma)=(0.1,0.03),\rho=0.8$ for fixed $\delta=0.4$ and $\eta=1.4$. We suggested a controller for fractional-order chaotic system (2) via fractional-order derivative based on the method due to Diethelm and Ford. The existence and uniqueness are established in Section 6, by imposing the sufficient condition on the coefficients of the system. The complex variables are taken in the unit disk. From above, we debated the fractional-order systems of complex variable and we pointed that they are more suitable than integer-order ones in biological, economic, and social systems. These

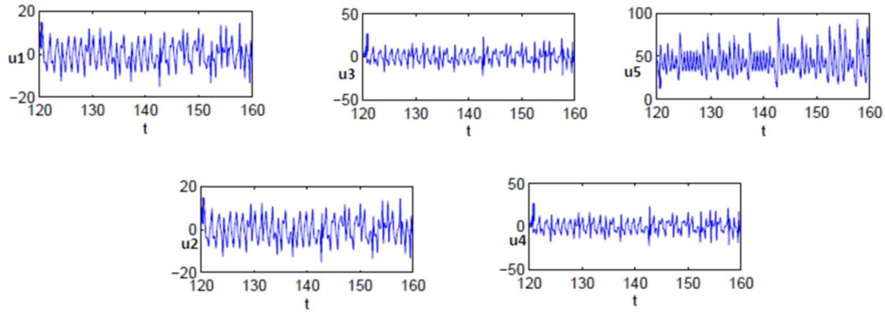

Figure 1. Time series for system (2).

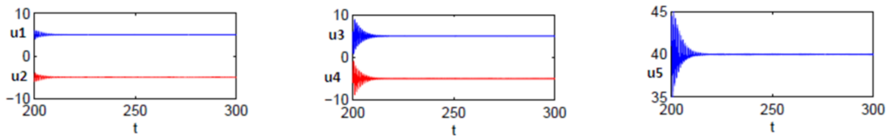

Figure 2. Waveform of system (2).

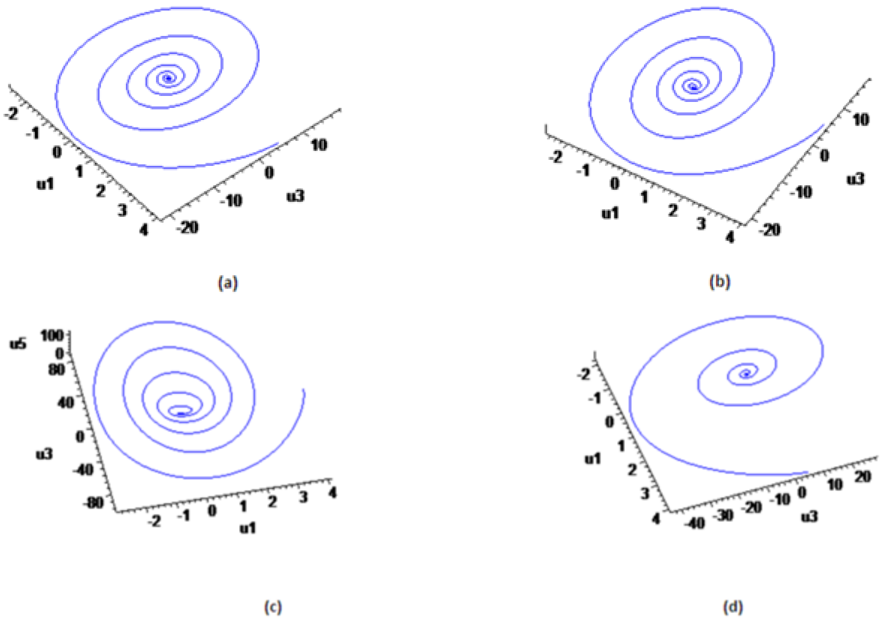

Figure 3. Stability cases for $\sigma > 0$, $\gamma > 0$.

complex systems, of five equations, are applied to characterize the physics of a laser, rotating fluids, disk dynamos, electronic circuits, and particle beam dynamics in high energy accelerators.

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