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Original Article

A new mixed beta distribution and structural properties with applications

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Abstract

In this paper, we introduce a new six-parameter distribution, namely Beta Exponentiated Weibull Poisson (BEWP) which is obtained by compounding between the exponentiated Weibull Poisson and beta distributions. We propose its basic structural properties such as density function and moments for this new distribution. We re-express the BEWP density function as a EWP linear combination, and use this to obtain its moments. In addition, it also contains several sub-models that are well known. Moreover, we apply the maximum likelihood method to estimate parameters, and applications to real data sets show the superiority of this new distribution by comparing the fitness with its sub-models.

Keywords: beta exponentiated weibull poisson, beta-g distribution

1. Introduction

For more than a decade, Weibull distribution has been applied extensively in many areas and more particularly used in the analysis of lifetime data for reliability engineering or biology (Rinne, 2008). However, the Weibull distribution has a weakness for modeling phenomenon with non-monotone failure rate. Therefore Mudholkar and Srivastava (1993) proposed the exponentiated Weibull (EW) distribution that is an extension of the Weibull family, obtained by adding a second shape parameter. Then it is flexible to model survival data where the failure rate can be increasing, decreasing, bathtub shape, or unimodal (Mudholkar *et al.*, 1995).

Let W be a random variable of the EW distribution. Then the cumulative distribution function (cdf) and probability density function (pdf) of W are given by

$$F(w) = \left(1 - e^{-(\theta w)^{\beta}}\right)^{\alpha}, \qquad w > 0,$$

where $\alpha, \beta, \theta > 0$ and

$$f(w) = \alpha\beta\theta^{\beta} w^{\beta-1} e^{-(\theta w)^{\beta}} \left(1 - e^{-(\theta w)^{\beta}}\right)^{\alpha-1}, \quad w > 0,$$

respectively.

For survival analysis, there are important analytical functions such as survival and hazard rate functions given by

$$S(w) = 1 - \left(1 - e^{-(\theta w)^{\beta}}\right)^{\alpha}$$

and

$$h(w) = \alpha\beta\theta^{\beta} w^{\beta-1} e^{-(\theta w)^{\beta}} \left(1 - e^{-(\theta w)^{\beta}}\right)^{\alpha-1} \left(1 - \left(1 - e^{-(\theta w)^{\beta}}\right)^{\alpha}\right)^{-1}.$$

respectively.

Recently, many researchers have attempted to modify EW distribution with different techniques by using EW as the baseline distribution to develop more flexibility. Pinho *et al.* (2012) proposed gamma exponentiated Weibull, Singla *et al.* (2012) studied beta generalized Weibull (BGW), Cordeiro *et al.* (2013) introduced the beta exponentiated Weibull (BEW) and exponentiated Weibull Poisson (EWP)

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was proposed by Mahmoudi and Sepahdar (2013).

In this paper, we propose a new flexible six-parameter distribution called Beta Exponentiated Weibull Poisson (BEWP) distribution. The purpose of this study is to create a new distribution by mixing EWP distribution and the beta distribution. Some properties of this new distribution will be investigated.

The BEWP distribution is developed by using the Beta-G distribution class that was introduced by Eugene *et al.* (2002) who also proposed the beta normal (BN) distribution. Then, using the Beta-G distribution class was applied to create a new distribution extensively. For example, Nadarajah and Gupta (2004) proposed the beta Frechet (BF) distribution and Nadarajah and Kotz (2004) studied the beta Gumbel (BGu) distribution. Nadarajah and Kotz (2006) introduced the beta exponential (BE) distribution, Lee *et al.* (2007) proposed the Beta Weibull (BW), Mahmoudi (2011) proposed the Beta Generalized Pareto (BGP), BGW or BEW and Percontini *et al.* (2013) studied the Beta Weibull Poisson (BWP).

The EWP distribution fits the skewed data (Mahmoudi and Sepahdar, 2013) and it is useful for solving complementary risks problem (Basu and Klein, 1982) in the presence of latent risks, in the sense that there is no information about which factor is responsible for the component failure and only the maximum lifetime value among all risks is observed. Mixing the EWP distribution with the beta distribution causes the two additional shape parameters which serve to control skewness and tail weights of EWP distribution. As a result, BEWP distribution is the generalized distribution that has a wide variety in terms of shape of the distribution, so it is a flexible alternative for applications in engineering and biology. In engineering applications, the BEWP distribution can be employed in reliability analysis, such as product reliability and system reliability. Percontini *et al.* (2013) applied the BWP distribution to the maintenance data on active repair times for airborne communication. In addition, for biology or medical science, we may apply to survival analysis e.g. Mudholkar *et al.* (1996) applied the generalized Weibull distribution in fitting the real survival time data of the patients who were given radiation therapy and chemotherapy from head and neck cancer clinical trial. Dasgupta *et al.* (2010) studied the characteristics of coronary artery calcium which is a marker of coronary artery disease. This appears to be a Weibull distribution and a Weibull regression model was proposed to examine factors influencing the disease. Ortega *et al.* (2013) developed the beta Weibull distribution to be the log-beta Weibull distribution and studied the log-beta Weibull regression model with application to predict recurrence of prostate cancer.

The rest of this paper is organized in the following sequence. Section 2 discusses about the exponentiated Weibull Poisson (EWP) distribution that is used as the baseline to develop the BEWP distribution. The probability density function (pdf) and cumulative density function (cdf) are introduced in Section 3. Section 4 gives a summary of sub-models of BEWP in the form of table and chart where several sub-models are well known. Section 5 discusses the moment generating function (mgf) and the moment. In Section 6 we apply the maximum likelihood method to estimate parameters, and Section 7 compares the sub-models of the BEWP distribution by the applications to real data sets. Some concluding remarks are given in Section 8.

2. The Exponentiated Weibull Poisson distribution

Let $W_1, W_2, W_3, ..., W_z$ be independent and identically distributed random variables from exponentiated Weibull distribution with pdf

$$f(w;\alpha,\beta,\theta) = \alpha\beta\theta^{\beta} w^{\beta-1} e^{-(\theta w)^{\beta}} \left(1-e^{-(\theta w)^{\beta}}\right)^{\alpha-1}, \quad w>0,$$

and Z, which is independent from *W's*, be a random variable from zero truncated Poisson distribution with probability mass function (pmf)

$$p(z;\lambda) = e^{-\lambda}\lambda^{z}\Gamma^{-1}(z+1)\left(1-e^{-\lambda}\right)^{-1}, \ \lambda>0, \ z=1,2,3,...$$

where $\Gamma(\bullet)$ is the gamma function. Percontini *et al.* (2013) described the model for $X = \min\{W_1, W_2, ..., W_z\}$ and $X = \max\{W_1, W_2, ..., W_z\}$ that can be used in serial and parallel system with identical components, which appear in many industrial applications and biological organisms. For this model, we define $X = \max\{W_1, W_2, ..., W_z\}$. We assume the failure occurs after all Z factors have been activated. Then we obtain

$$g(x|z;\alpha,\beta,\theta) = z\alpha\beta\theta^{\beta} w^{\beta-1} u(1-u)^{\alpha-1}(1-u)^{\alpha(z-1)}, \quad x>0,$$

where $u = e^{-(\theta x)^{\beta}}$ and the marginal pdf of X is

$$g(x) = \frac{\lambda\alpha\beta\theta^{\beta}}{\left(e^{\lambda}-1\right)} x^{\beta-1} u(1-u)^{\alpha-1} e^{\lambda(1-u)^{\alpha}}, \quad x>0, \tag{1}$$

and the cdf of X is

$$G(x) = \frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}. \tag{2}$$

Then we take this pdf and cdf of EWP to be the baseline for creating the new Beta-G distribution in the next section. We apply the interpretation of the EWP from Adamidis and Loukas (1998), that the failure (of a device, for example) occurs due to the presence of an unknown number, Z, of initial defects of the same kind (a number of semiconductors from a defective lot, for instance). The *W's* represent their lifetimes and each defect can be detected only after causing failure, in which case it is repaired perfectly.

3. The Beta Exponentiated Weibull Poisson distribution

Definition 1:

Let $F(x)$ be the cdf of a random variable X. According to Eugene *et al.* (2002), the cdf for a generalized class of distributions can be defined as the logit of beta random variable by

$$F\left(x\right)=\frac{1}{B\left(a,b\right)}\int_{0}^{G(x)} w^{a-1}\left(1-w\right)^{b-1}dw, \quad x>0, \tag{3}$$

where *a,b>0*. *G(x)* denotes the probability distribution of the generalized class of distribution. The pdf of X is given by (for more details see Mood *et al.* (1974))

$$f\left(x\right)=\frac{1}{B\left(a,b\right)}g\left(x\right)G\left(x\right)^{a-1}\left\{1-G\left(x\right)\right\}^{b-1}, \quad x>0, \tag{4}$$

where $g\left(x\right)=\dfrac{dG\left(x\right)}{dx}$.

Theorem 1:

Let *X* be a random variable of the BEWP distribution with parameters $\alpha,\beta,\theta,\lambda,a$ and *b*. The pdf of *X* is defined by

$$f\left(x\right)=\frac{\lambda\alpha\beta\theta^{\beta}x^{\beta-1}u\left(1-u\right)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}}{\left(e^{\lambda}-1\right)B\left(a,b\right)}\left(\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{a-1}\left(1-\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{b-1} \tag{5}$$

where $u=e^{-(\theta x)^{\beta}}$

Proof:

Simply by using Definition 1, we obtain the pdf of *X* by substituting *g(x)* and *G(x)* from Eqs.(1) and (2) into Eq.(4). Then Eq.(5) is the pdf of BEWP distribution as the following property.

$$\int_{0}^{\infty}f\left(x\right)dx=\int_{0}^{\infty}\frac{\lambda\alpha\beta\theta^{\beta}x^{\beta-1}u\left(1-u\right)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}}{\left(e^{\lambda}-1\right)B\left(a,b\right)}\left(\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{a-1}\left(1-\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{b-1}dx$$

let $v=e^{\lambda(1-u)^{\alpha}}$, it can be rewritten as

$$\int_{0}^{\infty}f\left(x\right)dx=\int_{1}^{e^{\lambda}}\frac{1}{\left(e^{\lambda}-1\right)B\left(a,b\right)}\left(\frac{v-1}{e^{\lambda}-1}\right)^{a-1}\left(1-\frac{v-1}{e^{\lambda}-1}\right)^{b-1}dv$$
$$=1.$$

Note that, we can define the expansion of the pdf as the linear combination of EWP density function as

$$f\left(x\right)=\sum_{j=0}^{\infty}\sum_{i=o}^{j}s_{j,i}g(x;\alpha,\beta,\theta,\lambda_{j,i}) \tag{6}$$

where $s_{j,i}=\dfrac{s_{j}(j+1)(-1)^{i}\binom{j}{i}\left(e^{\lambda_{j,i}}-1\right)}{(j-i+1)\left(e^{\lambda}-1\right)^{j+1}}$ and $s_{j}=\dfrac{r_{j+1}}{B\left(a,b\right)}$.

Let b be a non-integer real number and $\left|w\right|<1$.

$$F\left(x\right)=\frac{1}{B\left(a,b\right)}\int_{0}^{G(x)}w^{a-1}\left(1-w\right)^{b-1}dw, \quad a,b,x>0$$

By using the special case of binomial theorem

$$\left(1-w\right)^{b-1} = \sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}w^{i},$$

hence

$$F\left(x\right)=\frac{1}{B\left(a,b\right)}\int_{0}^{G(x)}w^{a-1}\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}w^{i}dw$$

$$=\frac{1}{B\left(a,b\right)}\sum_{i=0}^{\infty}\frac{(-1)^{i}\binom{b-1}{i}G\left(x\right)^{a+i}}{\left(a+i\right)}$$

$$=\frac{1}{B\left(a,b\right)}\sum_{i=0}^{\infty}c_{i}(a,b)G\left(x\right)^{a+i}, \tag{7}$$

where $c_{i}\left(a,b\right)=\dfrac{(-1)^{i}\binom{b-1}{i}}{\left(a+i\right)}$. If *b* is an integer, the index *i* stops at *b*-1. We obtain *f(x)* for integer *a*

$$f\left(x\right)=\frac{g(x)}{B\left(a,b\right)}\sum_{i=0}^{\infty}c_{i}\left(a,b\right)(a+i)G\left(x\right)^{a+i-1}.$$

Consider the case where the power of G(x) is a non-integer

$$G\left(x\right)^{\alpha}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha}{i}(1-G\left(x\right))^{i}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^{i+j}\Gamma\left(\alpha+1\right)G\left(x\right)^{j}}{\Gamma\left(\alpha-i+1\right)\left(i-j\right)!\,j!}$$

$$=\sum_{j=0}^{\infty}d_{j}(\alpha)G\left(x\right)^{j},$$

where $d_{j}(\alpha)=\sum_{j=i}^{\infty}\dfrac{(-1)^{i+j}\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha-i+1\right)\left(i-j\right)!\,j!}$, then we substitute $G\left(x\right)^{\alpha}$ for $G\left(x\right)^{a+i}$ in Eq. (7), we obtain

$$F\left(x\right)=\frac{1}{B\left(a,b\right)}\sum_{j=0}^{\infty}r_{j}(a,b)G\left(x\right)^{j}$$

and $$f\left(x\right)=\frac{1}{B\left(a,b\right)}g(x)\sum_{j=0}^{\infty}(j+1)r_{j}(a,b)G\left(x\right)^{j}$$

where $r_{j}\left(a,b\right)=\sum_{i=0}^{\infty}c_{i}(a,b)d_{j}(a+i)$ or

$$f\left(x\right)=\sum_{j=0}^{\infty}s_{j}(j+1)g(x)G\left(x\right)^{j},$$

where $s_{j}=\dfrac{r_{j+1}}{B\left(a,b\right)}$ and we can express pdf of BEWP distribution in terms of a linear combination of EWP distribution as

$$f\left(x\right)=\sum_{j=0}^{\infty}s_{j}(j+1)\left(\frac{\lambda\alpha\beta\theta^{\beta}x^{\beta-1}u\left(1-u\right)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}}{e^{\lambda}-1}\right)\left(\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{j}$$

$$=\sum_{j=0}^{\infty}\sum_{i=o}^{j}\frac{s_j(j+1)(-1)^i\binom{j}{i}}{(j-i+1)\left(e^{\lambda}-1\right)^{j+1}}\left(\lambda(j-i+1)\alpha\beta\theta^{\beta}x^{\beta-1}u\left(1-u\right)^{\alpha-1}e^{\lambda(1-u)^{\alpha}(j-i+1)}\right)$$

$$=\sum_{j=0}^{\infty}\sum_{i=o}^{j}\frac{s_j(j+1)(-1)^i\binom{j}{i}\left(e^{\lambda_{j,i}}-1\right)}{(j-i+1)\left(e^{\lambda}-1\right)^{j+1}}\left(\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}x^{\beta-1}u\left(1-u\right)^{\alpha-1}e^{\lambda_{j,i}(1-u)^{\alpha}}}{e^{\lambda_{j,i}}-1}\right)$$

$$=\sum_{j=0}^{\infty}\sum_{i=o}^{j}s_{j,i}g(x;\alpha,\beta,\theta,\lambda_{j,i})$$

where $\lambda_{j,i}=\lambda\left(j-i+1\right)$ and $s_{j,i}=\dfrac{s_j(j+1)(-1)^i\binom{j}{i}\left(e^{\lambda_{j,i}}-1\right)}{(j-i+1)\left(e^{\lambda}-1\right)^{j+1}}$

To show the various of shapes of the distribution, some specified parameters of the BEWP distribution and their density functions are provided in Figure 1: (a) Fix parameters $\alpha=4$, $\beta=0.5$, $\theta=1.5$, $\lambda=0.5$ and vary parameters a and b, and (b) Fix parameters $a=2$, $b=2$ and vary parameters α, β, θ and λ. Thus, the BEWP distribution can be suitable for fitting to various shapes of data, for example where the parameters $a=2$, $b=2$, $\alpha=4$, $\beta=0.5$, $\theta=1.5$, $\lambda=0.5$. This distribution is suitable for fitting skewed data and it is suitable for fitting unimodal data when the parameters $a=2$, $b=2$, $\alpha=0.5$, $\beta=2$, $\theta=0.1$, $\lambda=15$. According to Figure 1, BEWP distribution can be a family of distributions containing 32 sub-models which will be discussed in Section 4.

Theorem 2:

Let X be a random variable of a BEWP distribution with parameters $\alpha,\beta,\theta,\lambda,a$ and b. The cdf of X is given by

$$F\left(x\right)=\frac{1}{B\left(a,b\right)}\int_{0}^{(e^{\lambda(1-u)^{\alpha}}-1)/(e^{\lambda}-1)}w^{a-1}\left(1-w\right)^{b-1}dw \quad (8)$$

or

$$F\left(x\right)=I_{(e^{\lambda(1-u)^{\alpha}}-1)/(e^{\lambda}-1)}(a,b) \quad (9)$$

where $u=e^{-(\theta x)^{\beta}}$

Proof:

Simply by using Definition 1 again, we can define the cdf of X by replacing *G(x)* from Eq.(2) in Eq.(3), hence the cdf of BEWP distribution is as obtained in Eq.(8).

Note that, we can define the expansion of the cdf as the linear combination of EWP density function given by

$$F(x)=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}G(x;\alpha,\beta,\theta,\lambda_{j,i}) \quad (10)$$

by integrating *f(x)* in Eq.(6)

$$F(x)=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\int_{0}^{x}g(x;\alpha,\beta,\theta,\lambda_{j,i})$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\int_{0}^{x}\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}x^{\beta-1}u\left(1-u\right)^{\alpha-1}e^{\lambda_{j,i}(1-u)^{\alpha}}}{e^{\lambda_{j,i}}-1}dx$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\left(\frac{e^{\lambda_{j,i}(1-u)^{\alpha}}-1}{e^{\lambda_{j,i}}-1}\right)$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}G(x;\alpha,\beta,\theta,\lambda_{j,i})$$

4. Sub-models

This new distribution consists of a total of 32 sub-distribution models as shown in Figure 2 and Table 1. In Table 1, the sub-models associated with Poisson distribution, are assigned $X=\max\left\{W_1,W_2,...,W_z\right\}$ which based on the parallel components system comply with BEWP distribution that is under the same assumption. For $X=\min\left\{W_1,W_2,...,W_z\right\}$, we also refer to the References column and mark with the asterisk symbol (*) in Table 1.

(a) (b)

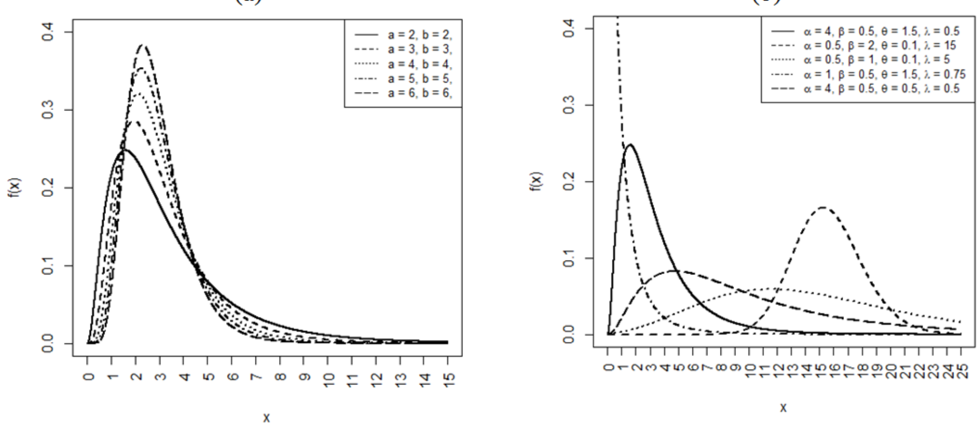

Figure 1. Density function of the BEWP distribution.

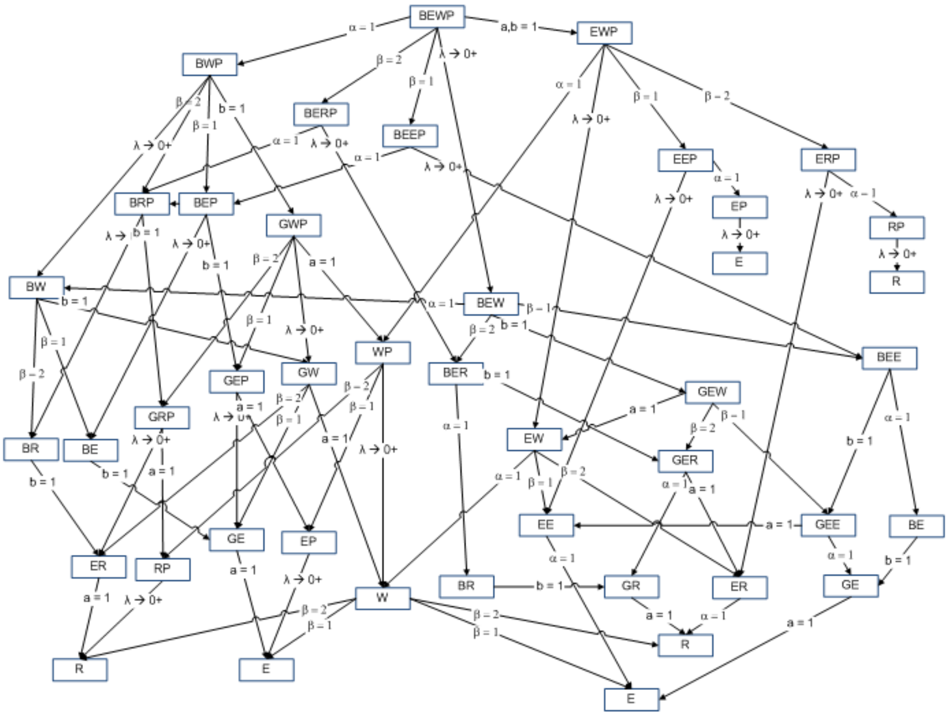

Figure 2. The sub-model chart of BEWP distribution

Table 1. The sub-model table of BEWP distribution.

Distribution	parameters						$F(x)$	References
	a	b	α	β	θ	λ		
5 parameters								
1. Beta Exponentiated Rayleigh Poisson (BERP)	a	b	α	2	θ	λ	$F(x)=I_{\frac{e^{\lambda(1-e^{-(\theta x)^2})^{\alpha}}-1}{e^{\lambda}-1}}(a,b)$	
2. Beta Exponentiated Exponential Poisson (BEEP)	a	b	α	1	θ	λ	$F(x)=I_{\frac{e^{\lambda(1-e^{-(\theta x)})^{\alpha}}-1}{e^{\lambda}-1}}(a,b)$	
3. Beta Weibull Poisson (BWP)	a	b	1	β	θ	λ	$F(x)=I_{\frac{e^{\lambda(1-e^{-(\theta x)^{\beta}})}-1}{e^{\lambda}-1}}(a,b)$	Percontini *et al.* (2013)*
4. Beta Exponentiated Weibull (BEW)	a	b	α	β	θ	→0	$F(x)=I_{(1-e^{-(\theta x)^{\beta}})^{\alpha}}(a,b)$	Singla *et al.* (2012), Cordeiro *et al.* (2013b)
4 parameters								
5. Beta Rayleigh Poisson (BRP)	a	b	1	2	θ	λ	$F(x)=I_{\frac{e^{\lambda(1-e^{-(\theta x)^2})}-1}{e^{\lambda}-1}}(a,b)$	
6. Beta Exponential Poisson (BEP)	a	b	1	1	θ	λ	$F(x)=I_{\frac{e^{\lambda(1-e^{-(\theta x)})}-1}{e^{\lambda}-1}}(a,b)$	
7. Generalized Weibull Poisson (GWP)	a	1	1	β	θ	λ	$F(x)=\left(\frac{e^{\lambda(1-e^{-(\theta x)^{\beta}})}-1}{e^{\lambda}-1}\right)^{a}$	
8. Exponentiated Weibull Poisson (EWP)	1	1	α	β	θ	λ	$F(x)=\frac{e^{\lambda(1-e^{-(\theta x)^{\beta}})^{\alpha}}-1}{e^{\lambda}-1}$	Mahmoudi and Sepahdar (2013)

Table 1. Continued

Distribution	parameters						F(x)	References
	a	b	α	β	θ	λ		
9. Generalized Exponentiated Weibull (GEW)	a	1	α	β	θ	→0	$F(x)=\left(1-e^{-(\theta x)^{\beta}}\right)^{\alpha a}$	
10. Beta Exponentiated Rayleigh (BER)	a	b	α	2	θ	→0	$F(x)=I_{(1-e^{-(\theta x)^{2}})^{\alpha}}(a,b)$	Cordeiro *et al.* (2013a)
11. Beta Exponentiated Exponential (BEE)	a	b	α	1	θ	→0	$F(x)=I_{\left(1-e^{-(\theta x)}\right)^{\alpha}}(a,b)$	Barreto-Souza *et al.* (2010)
12. Beta Weibull (BW)	a	b	1	β	θ	→0	$F(x)=I_{\left(1-e^{-(\theta x)^{\beta}}\right)}(a,b)$	Lee *et al.* (2007)
3 parameters								
13. Generalized Rayleigh Poisson (GRP)	a	1	1	2	θ	λ	$F(x)=\left(\frac{e^{\lambda(1-e^{-(\theta x)^{2}})}-1}{e^{\lambda}-1}\right)^{a}$	
14. Generalized Exponential Poisson (GEP)	a	1	1	1	θ	λ	$F(x)=\left(\frac{e^{\lambda(1-e^{-(\theta x)})}-1}{e^{\lambda}-1}\right)^{a}$	Barreto-Souza, and Cribari-Neto (2009)
15. Exponentiated Rayleigh Poisson (ERP)	1	1	α	2	θ	λ	$F(x)=\frac{e^{\lambda(1-e^{-(\theta x)^{2}})^{\alpha}}-1}{e^{\lambda}-1}$	Mahmoudi and Sepahdar (2013)
16. Exponentiated Exponential Poisson (EEP)	1	1	α	1	θ	λ	$F(x)=\frac{e^{\lambda(1-e^{-(\theta x)})^{\alpha}}-1}{e^{\lambda}-1}$	Percontini *et al.* (2013)*, Ristiæ and Nadarajah (2014)*, Mahmoudi and Sepahdar (2013)
17. Weibull Poisson (WP)	1	1	1	β	θ	λ	$F(x)=\frac{e^{\lambda(1-e^{-(\theta x)^{\beta}})}-1}{e^{\lambda}-1}$	Hemmati *et al.* (2011)*, Lu and Shi (2012)*, Mahmoudi and Sepahdar (2013)
18. Exponentiated Weibull (EW)	1	1	α	β	θ	→0	$F(x)=(1-e^{-(\theta x)^{\beta}})^{\alpha}$	Mudolkar and Srivastava (1993) coincide with GW
19. Generalized Weibull(GW)	a	1	1	β	θ	→0	$F(x)=\left(1-e^{-(\theta x)^{\beta}}\right)^{a}$	Mudolkar and Srivastava (1993) coincide with EW
20. Generalized Exponentiated Rayleigh(GER)	a	1	α	2	θ	→0	$F(x)=\left(1-e^{-(\theta x)^{2}}\right)^{\alpha a}$	Cordeiro *et al.* (2013a)
21. Generalized Exponentiated Exponential(GEE)	a	1	α	1	θ	→0	$F(x)=\left(1-e^{-(\theta x)}\right)^{\alpha a}$	
22. Beta Rayleigh (BR)	a	b	1	2	θ	→0	$F(x)=I_{\left(1-e^{-(\theta x)^{2}}\right)}(a,b)$	
23. Beta Exponential(BE)	a	b	1	1	θ	→0	$F(x)=I_{\left(1-e^{-(\theta x)}\right)}(a,b)$	Nadarajah and Kotz (2006)
2 parameters								
24. Rayleigh Poisson (RP)	1	1	1	2	θ	λ	$F(x)=\frac{e^{\lambda(1-e^{-(\theta x)^{2}})}-1}{e^{\lambda}-1}$	Mahmoudi and Sepahdar (2013)
25. Exponential Poisson (EP)	1	1	1	1	θ	λ	$F(x)=\frac{e^{\lambda(1-e^{-(\theta x)})}-1}{e^{\lambda}-1}$	Kus (2007)*, Cancho *et al.* (2011)
26. Weibull (W)	1	1	1	β	θ	→0	$F(x)=(1-e^{-(\theta x)^{\beta}})$	Mudolkar and Srivastava (1993)

Table 1. Continued

Distribution	parameters						$F(x)$	References
	a	b	α	β	θ	λ		
27. Exponentiated Rayleigh (ER)	1	1	α	2	θ	→0	$F(x)=\left(1-e^{-(\theta x)^2}\right)^{\alpha}$	Kundu and Raqab (2005) coincide with GR
28. Generalized Rayleigh (GR)	a	1	1	2	θ	→0	$F(x)=\left(1-e^{-(\theta x)^2}\right)^{a}$	Kundu and Raqab (2005) coincide with ER
29. Exponentiated Exponential (EE)	1	1	α	1	θ	→0	$F(x)=(1-e^{-(\theta x)})^{\alpha}$	Gupta and Kundu (1999) coincide with GE
30. Generalized Exponential (GE)	a	1	1	1	θ	→0	$F(x)=\left(1-e^{-(\theta x)}\right)^{a}$	Gupta and Kundu (1999) coincide with EE
31. Rayleigh(R)	1	1	1	2	θ	→0	$F(x)=\left(1-e^{-(\theta x)^2}\right)$	Johnson *et al.* (1994)
32. Exponential(E)	1	1	1	1	θ	→0	$F(x)=\left(1-e^{-(\theta x)}\right)$	Johnson *et al.* (1994)

5. Moment Generating Function and Moment

Theorem 3:

Let X be a random variable of a BEWP distribution with parameters $\alpha,\beta,\theta,\lambda,a$ and b. The moment generating function (mgf) of X can be given by

$$M_X(t)=\sum_{m=0}^{\infty}\omega_m t^m \tag{11}$$

where $\omega_m=\sum_{j,l,n=0}^{\infty}\sum_{i=0}^{j}\upsilon(i,j,l,m,n)\Gamma\left(\frac{m}{\beta}+1\right)$ and $m=0,1,2,\ldots$

Proof:

To find the mgf of BEWP distribution, we apply the definition of mgf to the linear combination of EWP density function as

$$M_{X_{BEWP}}(t)=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\int_0^{\infty}e^{tx}g(x;\alpha,\beta,\theta,\lambda_{j,i})dx$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}M_{X_{EWP}}(t;\alpha,\beta,\theta,\lambda_{j,i}),$$

where Mahmoudi and Sepahdar (2013) derived that

$$M_{X_{EWP}}(t)=\frac{\lambda_{j,i}\alpha}{e^{\lambda_{j,i}}-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\lambda_{j,i}^{n}}{n!}\frac{t^m\theta^{-m}}{m!}\Gamma\left(\frac{m}{\beta}+1\right)(-1)^l\binom{\alpha(n+1)-1}{l}(l+1)^{-\left(\frac{m}{\beta}+1\right)}.$$

We can reduce to

$$M_{X_{BEWP}}(t)=\sum_{j,l,m,n=0}^{\infty}\sum_{i=0}^{j}\upsilon(i,j,l,m,n)\Gamma\left(\frac{m}{\beta}+1\right)t^m,$$

where

$$\upsilon(i,j,l,m,n)=\frac{\alpha s_{j,i}\lambda_{j,i}^{n+1}\theta^{-m}}{\left(e^{\lambda_{j,i}}-1\right)n!m!}(-1)^l\binom{\alpha(n+1)-1}{l}(l+1)^{-\left(\frac{m}{\beta}+1\right)}.$$

And we can reduce again to be

$$M_{X_{BEWP}}(t)=\sum_{m=0}^{\infty}\omega_m t^m$$

where $\omega_m=\sum_{j,l,n=0}^{\infty}\sum_{i=0}^{j}\upsilon(i,j,l,m,n)\Gamma\left(\frac{m}{\beta}+1\right)$ and $m=0,1,2,\ldots$

Theorem 4:

Let X be a random variable of a BEWP distribution with parameters $\alpha,\beta,\theta,\lambda,a$ and b. The moment of X can be written as

$$E(X^m)=\alpha\theta^{-m}\Gamma\left(\frac{m}{\beta}+1\right)\sum_{j,n,l=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}^{n+1}}{\left(e^{\lambda_{j,i}}-1\right)n!}(-1)^l\binom{\alpha(n+1)-1}{l}(l+1)^{-\left(\frac{m}{\beta}+1\right)} \tag{12}$$

Proof:

To find the moment of BEWP distribution, we apply the definition of moment again to the linear combination of EWP density function as

$$E_{BEWP}(X^m)=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\int_0^{\infty}x^m g(x;\alpha,\beta,\theta,\lambda_{j,i})dx$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}E_{EWP}(X^m;\alpha,\beta,\theta,\lambda_{j,i}),$$

where Mahmoudi and Sepahdar (2013) derived that

$$E_{EWP}(X^m)=\alpha\theta^{-m}\Gamma\left(\frac{m}{\beta}+1\right)\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{\lambda_{j,i}^{n+1}}{\left(e^{\lambda_{j,i}}-1\right)n!}(-1)^l\binom{\alpha(n+1)-1}{l}(l+1)^{-\left(\frac{m}{\beta}+1\right)}.$$

So we can obtain the moment of BEWP distribution as

$$E(X^m)=\alpha\theta^{-m}\Gamma\left(\frac{m}{\beta}+1\right)\sum_{j,n,l=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}^{n+1}}{\left(e^{\lambda_{j,i}}-1\right)n!}(-1)^l\binom{\alpha(n+1)-1}{l}(l+1)^{-\left(\frac{m}{\beta}+1\right)}.$$

Then we can find variance, skewness and kurtosis of random variable X by using the well-known relationship of each moment.

6. Parameter Estimation

In this section, we suppose that the sample size n was drawn from BEWP distribution, and let $\boldsymbol{\pi}=(\alpha,\beta,\theta,\lambda,a,b)^T$ be the parameter vector. Then the log-likelihood function of BEWP is given by

$$l(\alpha,\beta,\theta,\lambda,a,b)=\sum_{i=1}^{n}\Big(\log(\alpha)+\log(\lambda)+\log(\beta)+\beta\log(\theta)-\log\left[B(a,b)\right]$$
$$-(\theta x_i)^\beta+(\alpha-1)\log(1-u_i)+\lambda(1-u_i)+(a-1)\log\left(e^{\lambda(1-u_i)^\alpha}-1\right)$$
$$+(b-1)\log\left(e^{\lambda}-e^{\lambda(1-u_i)^\alpha}\right)-(a+b-1)\log\left(e^{\lambda}-1\right)\Big)$$

where $u_i=e^{-(\theta x_i)^\beta}$. So the elements of score vector

$$\boldsymbol{U}=\left(\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial\beta},\frac{\partial l}{\partial\theta},\frac{\partial l}{\partial\lambda},\frac{\partial l}{\partial a},\frac{\partial l}{\partial b}\right)^T$$

where

$$\frac{\partial l}{\partial\alpha}=\sum_{i=1}^{n}\left(\frac{1}{\alpha}+\log(1-u_i)+\frac{(a-1)\lambda(1-u_i)^\alpha e^{\lambda(1-u_i)^\alpha}\log(1-u_i)}{e^{\lambda(1-u_i)^\alpha}-1}\right.$$
$$\left.+\frac{-(b-1)\lambda(1-u_i)^\alpha e^{\lambda(1-u_i)^\alpha}\log(1-u_i)}{e^{\lambda}-e^{\lambda(1-u_i)^\alpha}}\right)$$

$$\frac{\partial l}{\partial\beta}=\sum_{i=1}^{n}\left(\frac{1}{\beta}+\log(\theta)-(\theta x_i)^\beta\left(\log(x_i)+\log(\theta)\right)+\frac{(\alpha-1)(\theta x_i)^\beta u_i\left(\log(x_i)+\log(\theta)\right)}{1-u_i}\right.$$
$$+\lambda(\theta x_i)^\beta u_i\left(\log(x_i)+\log(\theta)\right)$$
$$+\frac{(a-1)\alpha\lambda(\theta x_i)^\beta(1-u_i)^{\alpha-1}e^{\lambda(1-u_i)^\alpha-(\theta x_i)^\beta}\left(\log(x_i)+\log(\theta)\right)}{e^{\lambda(1-u_i)^\alpha}-1}$$
$$\left.+\frac{-(b-1)\alpha\lambda(\theta x_i)^\beta(1-u_i)^{\alpha-1}e^{\lambda(1-u_i)^\alpha-(\theta x_i)^\beta}\left(\log(x_i)+\log(\theta)\right)}{e^{\lambda}-e^{\lambda(1-u_i)^\alpha}}\right)$$

$$\frac{\partial l}{\partial\theta}=\sum_{i=1}^{n}\left(\frac{\beta}{\theta}-\frac{\beta(\theta x_i)^\beta}{\theta}+\frac{(\alpha-1)\beta(\theta x_i)^\beta u_i}{\theta(1-u_i)}+\frac{(a-1)\alpha\lambda\beta(\theta x_i)^\beta(1-u_i)^{\alpha-1}e^{\lambda(1-u_i)^\alpha-(\theta x_i)^\beta}}{\theta\left(e^{\lambda(1-u_i)^\alpha}-1\right)}\right.$$
$$\left.+\frac{\lambda\beta(\theta x_i)^\beta u_i}{\theta}+\frac{-(b-1)\alpha\lambda\beta(\theta x_i)^\beta(1-u_i)^{\alpha-1}e^{\lambda(1-u_i)^\alpha-(\theta x_i)^\beta}}{\theta\left(e^{\lambda}-e^{\lambda(1-u_i)^\alpha}\right)}\right)$$

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\left(\frac{1}{\lambda}-\frac{(a+b-1)e^{\lambda}}{e^{\lambda}-1}+(1-u_i)+\frac{(a-1)(1-u_i)^\alpha e^{\lambda(1-u_i)^\alpha}}{e^{\lambda(1-u_i)^\alpha}-1}\right.$$
$$\left.+\frac{(b-1)\left(e^{\lambda}-(1-u_i)^\alpha e^{\lambda(1-u_i)^\alpha}\right)}{e^{\lambda}-e^{\lambda(1-u_i)^\alpha}}\right)$$

$$\frac{\partial l}{\partial a}=\sum_{i=1}^{n}\left(-\left(\psi(a)-\psi(a+b)\right)+\log\left(e^{\lambda(1-u_i)^\alpha}-1\right)-\log\left(e^{\lambda}-1\right)\right)$$

$$\frac{\partial l}{\partial b}=\sum_{i=1}^{n}\left(-\left(\psi(b)-\psi(a+b)\right)+\log\left(e^{\lambda}-e^{\lambda(1-u_i)^\alpha}\right)-\log\left(e^{\lambda}-1\right)\right)$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. The maximum likelihood estimator $\hat{\boldsymbol{\pi}}=\left(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\lambda},\hat{a},\hat{b}\right)^T$ is the solution to the above score equations that are calculated by using Newton-Raphson method in R package (R Core Team, 2012).

7. Applications

In this section, to reveal the superiority of BEWP distribution, we fit a BEWP model to two real data sets from the application of EWP (Mahmoudi and Sepahdar, 2013). The first application, we study the skewed data representing strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England, which are given in Table 2. Unfortunately, the units of measurement are not given in the paper.

For the second application, we examine the data showing the stress-rupture life of Kevlar 49/epoxy strands (unit: hours) which were subjected to constant sustained pressure at the 90 stress level until all had failed as displayed in Table 3.

We fit the BEWP distribution to above two data sets and compare the fitness with its sub-models that are BWP, BEW, BEE, EWP, EW, EE including Weibull distribution by considering the p-value of Kolmogorov-Smirnov (K-S) statistics. The maximum likelihood estimates of the parameters, the K-S statistics and the corresponding p-value for the fitted models are shown for data sets I and II in Tables 3 and 4, respectively. Graphical approach is the another way to express these data sets fit with this distribution. We also present the comparison of the empirical cdf wtth each estimated cdf in Figure 3. It shows the fitting of data to proposed models.

The probability plots of the BEWP distribution corresponding to data sets I and II in Figure 4 indicate that (a) most data lie around the straight line especially the middle 50% of the data, and (b) the first 75% of the data lie on the straight line and the last 25% of the data lie above, which suggests a slight right-skewness. It seems reasonable to tentatively conclude that both data sets are BEWP distribution.

8. Conclusion

For this paper, a new six-parameter distribution, namely BEWP is studied. It is obtained by compounding beta

Table 2. Strengths of 1.5 cm glass fibers

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64
1.68	1.73	1.81	2.00	0.74	1.04	1.27	1.39	1.49
1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01	0.77
1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69
1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.50	1.55
1.61	1.62	1.66	1.70	1.77	1.84	0.84	1.24	1.30
1.48	1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89

Table 3. Stress-rupture life of Kevlar 49/epoxy strands (unit: hours)

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06
0.07	0.07	0.08	0.09	0.09	0.10	0.10	0.11	0.10	0.12
0.13	0.18	0.19	0.20	0.23	0.24	0.24	0.29	0.34	0.35
0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.60
0.63	0.65	0.67	0.68	0.72	0.72	0.72	0.73	0.79	0.79
0.80	0.80	0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29
1.31	1.33	1.34	1.40	1.43	1.45	1.50	1.51	1.52	1.53
1.54	1.54	1.55	1.58	1.60	1.63	1.64	1.80	1.80	1.81
2.02	2.05	2.14	2.17	2.33	3.03	3.03	3.34	4.20	4.69
7.89									

Table 4. MLE and K-S statistics with corresponding p-values for the strengths of 1.5 cm glass fibers.

Fitting Distribution	Parameters						K-S	p-value
	a	b	α	β	θ	λ		
BEWP	0.1203	0.4896	6.0391	4.9958	0.7693	12.2998	0.0705	0.9127
BWP	0.5032	0.8554	-	5.3956	0.6782	4.6781	0.0978	0.5827
BEW	0.3703	3.924	2.3156	5.6507	0.5135	-	0.1448	0.1425
BEE	0.4021	31.4853	24.8020	-	1.0976	-	0.1999	0.0130
EWP	-	-	0.5781	5.5015	0.6466	2.7821	0.1154	0.3713
EW	-	-	0.6712	7.2845	0.5820	-	0.1462	0.1351
EE	-	-	31.3485	-	2.6115	-	0.2291	0.0027
Weibull	-	-	-	5.7807	0.6142	-	0.1522	0.1078

and exponentiated Weibull Poisson distributions. We introduce its basic mathematical properties such as density function. We show that the pdf of BEWP distribution can be expressed in the linear combination form of EWP distribution including its moments. Moreover, it also contains the many sub-models that are well known. Finally, we have applied the maximum likelihood method to estimate parameters and fit the BEWP distribution to two real data sets. We compared the results with its sub-models such as BWP, BEW, BEE, EWP, EW, EE and Weibull distribution. The results showed that BEWP distribution provides a better fit than existing mixtures of the EW or Weibull distribution and some well-known sub-models.

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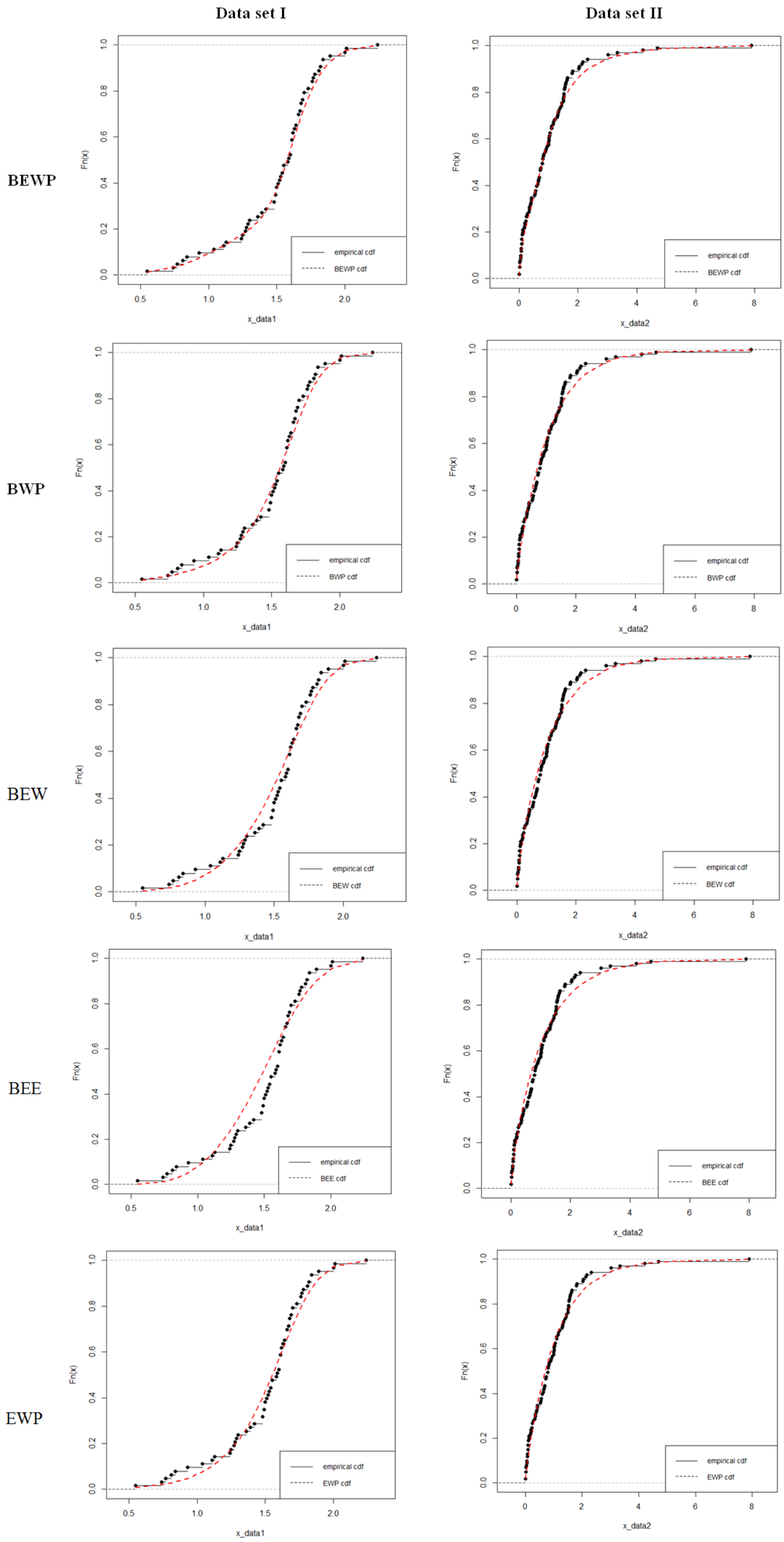

Figure 3. Comparison between empirial cdf and estimated cdf of data sets I and II.

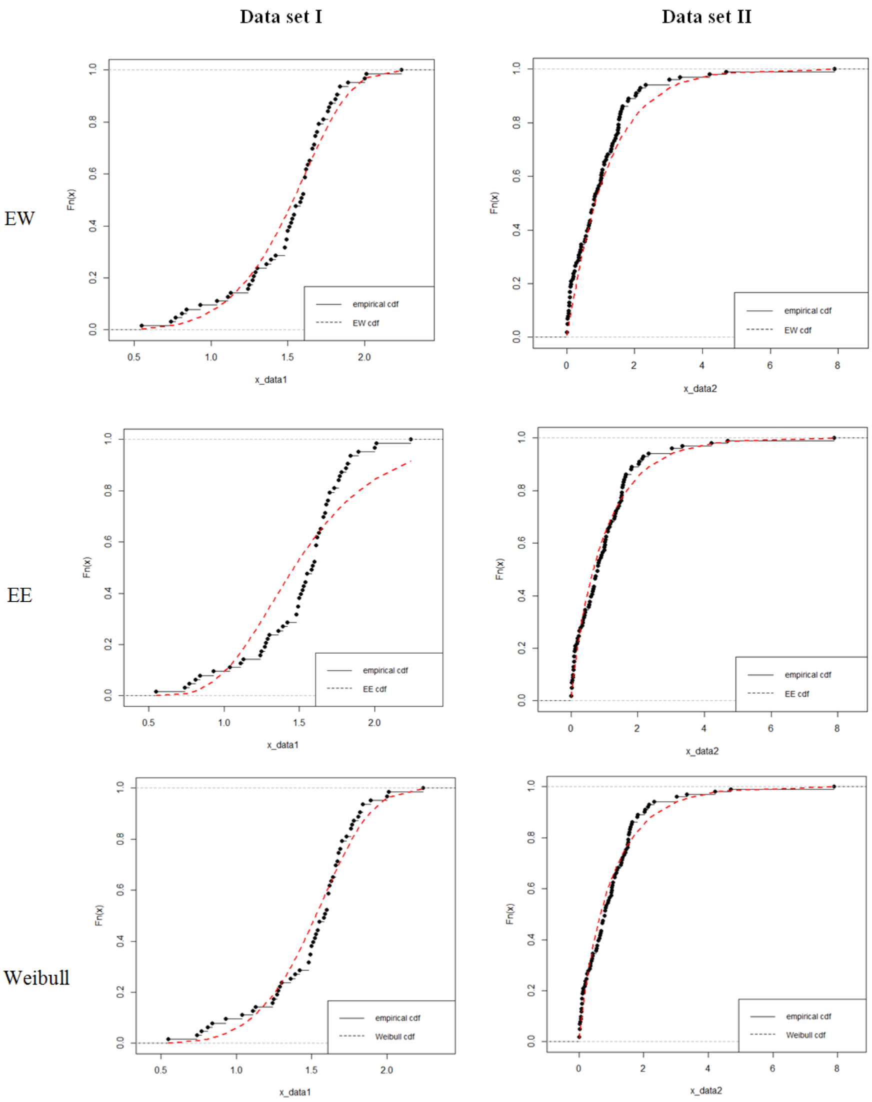

Figure 3. Comparison between empirial cdf and estimated cdf of data sets I and II. (Continued)

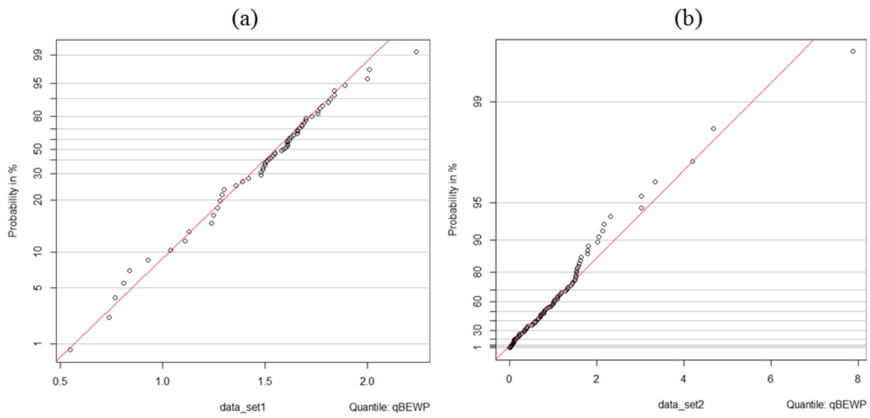

Figure 4. The probability plot of the BEWP distribution of data sets I and II.

Table 5 MLE and K-S statistics with corresponding p-values for the stress-rupture life of Kevlar 49/epoxy strands (unit: hours)

Fitting Distribution	Parameters						K-S	p-value
	a	b	α	β	θ	λ		
BEWP	0.1262	0.5730	15.8498	0.6127	8.2299	9.1019	0.0615	0.8399
BWP	0.8164	1.1569	-	0.8659	1.2109	1.7244	0.0666	0.7623
BEW	1.0218	0.3523	0.7023	1.0413	2.3771	-	0.0814	0.5145
BEE	0.5805	0.2222	1.5364	-	4.0599	-	0.0905	0.3801
EWP	-	-	0.8589	0.8717	1.3032	1.2662	0.0725	0.664
EW	-	-	0.9729	1.0604	0.821	-	0.0844	0.468
EE	-	-	0.8663	-	0.8883	-	0.0887	0.4044
Weibull	-	-	-	0.9259	1.0102	-	0.0906	0.3778

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