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Original Article

On (*m, n*)-ideals and (*m, n*)-regular ordered semigroups

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Abstract

Let *m, n* be non-negative integers. A subsemigroup A of an ordered semigroup $(S, \cdot, \leq$ is called an (*m, n*)-ideal of S if (*i*) $A^m SA^n \subseteq A$, and (*ii*) if $x \in A$, $y \in S$ such that $y \leq x$, then $y \in A$. In this paper, necessary and sufficient conditions for every (*m, n*)-ideal (resp. (*m, n*)-quasi-ideal) of an (*m, n*)-ideal (resp. (*m, n*)-quasi-ideal) A of S is an (*m, n*)-ideal (resp. (*m, n*)-quasi-ideal) of S will be given. Moreover, (*m, n*)-regularity of S will be discussed. The results obtained extend the results on semigroups (without order) studied by Bogdanovic′ (1979).

Keywords: semigroup, ordered semigroup, (*m, n*)-ideal, (*m, n*)-quasi-ideal, (*m, n*)-regular

1. Preliminaries

Let *m, n* be non-negative integers. A subsemigroup A of a semigroup S is called an (*m, n*)-ideal of S if

$$A^m SA^n \subseteq A.$$

Here, $A^0S = SA^0 = S$. This notion was first introduced and studied by Lajos (1961). Furthermore, the theory of (*m, n*)-ideals in other structures have also been studied by many authors (see also Akram *et al.*, 2013; Amjad *et al.*, 2014; Lajos, 1963; Yaqoob *et al.*, 2012; Yaqoob *et al.*, 2013; Yaqoob *et al.*, 2014; Yousafzai *et al.*, 2014). A semigroup S is said to be (*m, n*)-*regular* (Krgovic′, 1975) if for any a in S, there exists x in S such that $a = a^m xa^n$. Bogdanovic′ (1979) studied some properties of (*m, n*)-ideals and (*m, n*)-regularity of S. Indeed, the author characterized when every (*m, n*)-ideal of an (*m, n*)-ideal A of S is an (*m, n*)-ideal of S. Moreover, (*m, n*)-regularity of S was discussed. In this paper, using the concepts of (*m, n*)-ideals and (*m, n*)-regularity of ordered semigroups introduced and studied by Sanboorisoot *et al.* (2012), we extend the results obtained by Bogdanovic′ (1979) mentioned above to ordered semigroups.

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A semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that, for any *a, b, c* in S,

$$a \leq b \Rightarrow ac \leq bc, ca \leq cb,$$

is said to be an *ordered semigroup* (Birkhoff, 1967; Fuchs, 1963). A non-empty subset A of an ordered semigroup $(S, \cdot, \leq)$ is said to be a *subsemigroup* of S if $ab \in A$ for all a, b in A (Kehayopulu, 2006).

If A and B are non-empty subsets of an ordered semigroup $(S, \cdot, \leq)$, the set product AB is defined to be the set of all elements $ab \in S$ such that $a \in A$ and $b \in B$, that is, $AB = \{ab \mid a \in A, b \in B\}$. And, we write

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

It is observed by Kehayopulu (2006) that the following conditions hold: (1) $A \subseteq (A]$; (2) $(A](B] \subseteq (AB]$; (3) If $A \subseteq B$, then $(A] \subseteq (B]$; (4) $(A \cup B] = (A] \cup (B]$; (5) $(A \cap B] \subseteq (A] \cap (B]$.

A non-empty subset A of an ordered semigroup $(S, \cdot, \leq)$ is called *a left* (resp. *right*) *ideal* of S if it satisfies the following conditions: (i) $SA \subseteq A$ (resp. $AS \subseteq A$); (ii) $(A] = A$. And, A is called a *two-sided ideal* (or simply an *ideal*) of S if it is both a left and a right ideal of S (Kehayopulu, 2006). A subsemigroup B of S is called a *bi-ideal* of S if (i) $BSB \subseteq B$; (ii) $(B] = B$ (Kehayopulu, 1992). A non-empty subset Q of S is called a *quasi-ideal* of S if (i) $(QS]\cap(SQ]\subseteq Q$; (ii) $(Q]=Q$ (Tsingelis, 1991; Kehayopulu, 1994). Note that if Q is a quasi-ideal of S, then it is a subsemigroup of S. In fact, if Q is a quasi-ideal of S, then $QQ\subseteq(QS]\cap(SQ]\subseteq Q$. Finally, a subsemigroup A of S is called an (m, n)-ideal of S (m, n are non-negative integers) if (i) $A^mSA^n\subseteq A$; (ii) $(A]=A$ (Sanborisoot *et al.*, 2012).

We first prove the following theorem.

Theorem 1.1. Let A be a non-empty subset of an ordered semigroup $(S,\cdot,\leq)$. Then the intersection of all (m, n)-ideals containing A of S, denoted by $[A]_{(m,n)}$, is an (m, n)-ideal containing A of S, and it is of the form

$$[A]_{(m,n)}=\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big] \qquad (1.1)$$

Proof. Let $\{A_i \mid i\in I\}$ be the set of all (m, n)-ideals containing A of S. Then $\bigcap_{i\in I}A_i$ is a subsemigroup containing A of S. For $j\in I$, we have

$$\Big(\bigcap_{i\in I}A_i\Big)^mS\Big(\bigcap_{i\in I}A_i\Big)^n\subseteq A_j^mSA_j^n\subseteq A_j.$$

Then $(\bigcap_{i\in I}A_i)^mS(\bigcap_{i\in I}A_i)^n\subseteq\bigcap_{i\in I}A_i$. Since

$$\Big(\bigcap_{i\in I}A_i\Big]\subseteq\bigcap_{i\in I}(A_i]=\bigcap_{i\in I}A_i\subseteq\Big(\bigcap_{i\in I}A_i\Big],$$

it follows that $\bigcap_{i\in I}A_i$ is an (m, n)-ideal of S.

We will show that (1.1) holds. It is easy to see that $\big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\big]$ is a subsemigroup of S. We now consider:

$$\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^mS$$

$$=\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^{m-1}\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]S$$

$$\subseteq\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^{m-1}(AS]$$

$$=\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^{m-2}\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big](AS]$$

$$\subseteq\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^{m-2}(A^2S]$$

$$\vdots$$

$$\subseteq(A^mS].$$

Similarly,

$$S\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^n\subseteq(SA^n].$$

Then,

$$\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^mS\Big(\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\Big)^n$$

$$\subseteq(A^mSA^n]$$

$$\subseteq\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big].$$

Hence $\big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\big]$ is an (m, n)-ideal containing A of S, and

$$[A]_{(m,n)}\subseteq\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big].$$

Finally, by

$$\big(A^mSA^n\big]\subseteq\big(([A]_{(m,n)})^mS([A]_{(m,n)})^n\big]\subseteq[A]_{(m,n)},$$

it follows that

$$\Big(\bigcup_{i=1}^{m+n}A^i\cup A^mSA^n\Big]\subseteq[A]_{(m,n)}.$$

This completes the proof. □

For an element a of an ordered semigroup $(S,\cdot,\leq)$, we write $[\{a\}]_{(m,n)}$ (or simply $[a]_{(m,n)}$) by:

$$[a]_{(m,n)}=\Big(\bigcup_{i=1}^{m+n}\{a\}^i\cup a^mSa^n\Big].$$

To extend the notion of (m, n)-quasi-ideals of semigroups defined by Lajos (1961), we introduce the concept of (m, n)-quasi-ideals of an ordered semigroup $(S,\cdot,\leq)$ as follows: let m, n be non-negative integers. A subsemigroup Q of S is called an (m, n)-*quasi-ideal* of S if it satisfies the following conditions:

(i) $(Q^mS]\cap(SQ^n]\subseteq Q$;

(ii) $(Q]=Q$.

Here, $Q^0S=SQ^0=S$. Note that every (m, n)-quasi-ideal of S is an (m, n)-ideal of S.

It's easy to see that if Q is a quasi-ideal of S, then Q is an (m, n)-quasi-ideal of S. The following example shows that an (m, n)-quasi-ideal of S needs not to be a quasi-ideal of S.

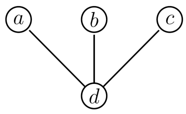
Example 1.1. Let $S=\{a,b,c,d\}$ be an ordered semigroup with the multiplication and the order relation defined by:

$\cdot$	a	b	c	d
a	d	c	d	d
b	c	d	d	d
c	d	d	d	d
d	d	d	d	d

$$\leq = \{(a,a),(b,b),(c,c),(d,a),(d,b),(d,c),(d,d)\}.$$

We give the covering relation and the figure of S by:

$$\prec = \{(d,a),(d,b),(d,c)\}$$



Let $Q = \{a, d\}$. For integers $m, n > 1$, we obtain that Q is an (m, n)-quasi-ideal of S but not a quasi-ideal of S.

As in Theorem 1.1, we have the following.

Theorem 1.2. Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the intersection of all (m, n)-quasi-ideals containing a non-empty subset A of S, denoted by $[A]_{q,(m,n)}$, is an (m, n)-quasi-ideal containing A of S, and it is of the form

$$[A]_{q,(m,n)} = \Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big). \quad (1.2)$$

Proof. Let $\{A_i \mid i \in I\}$ be the set of all (m, n)-quasi-ideals containing A of S. Then $\bigcap_{i\in I} A_i$ is a subsemigroup containing A of S. For $j \in I$, we have

$$((\bigcap_{i\in I} A_i)^m S] \cap (S(\bigcap_{i\in I} A_i)^n] \subseteq ((A_j)^m S] \cap (S(A_j)^n] \subseteq A_j,$$

and then $((\bigcap_{i\in I} A_i)^m S] \cap (S(\bigcap_{i\in I} A_i)^n] \subseteq \bigcap_{i\in I} A_i$. Moreover,

$$(\bigcap_{i\in I} A_i] \subseteq \bigcap_{i\in I} (A_i] = \bigcap_{i\in I} A_i \subseteq (\bigcap_{i\in I} A_i],$$

and hence $\bigcap_{i\in I} A_i$ is an (m, n)-quasi-ideal of S.

Next, we will show that (1.2) holds. Clearly, $(\bigcup_{i=1}^{\max\{m,n\}} A^i] \cup ((A^m S] \cap (SA^n]) \neq \varnothing$. Let $x, y \in (\bigcup_{i=1}^{\max\{m,n\}} A^i] \cup ((A^m S] \cap (SA^n])$. If $x \in (A^m S] \cap (SA^n]$ or $y \in (A^m S] \cap (SA^n]$, then

$$xy \in (A^m S] \cap (SA^n] \subseteq \Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big).$$

Let $x, y \in (\bigcup_{i=1}^{\max\{m,n\}} A^i]$; then there exist j, k in $\{1, 2, \ldots, \max\{m,n\}\}$ such that $x \in (A^j]$ and $y \in (A^k]$. If $1 < j + k \leq \max\{m,n\}$, then

$$xy \in \Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \subseteq \Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big).$$

If $\max\{m,n\} < j + k$, then $m, n < j + k$, that is, $(A^{j+k}] = (A^{m+(j+k-m)}] \subseteq (A^m S]$ and $(A^{j+k}] = (A^{(j+k-n)+n}] \subseteq (SA^n]$. Hence

$$xy \in (A^{j+k}] \subseteq (A^m S] \cap (SA^n] \subseteq \Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big).$$

This shows that $(\bigcup_{i=1}^{m+n} A^i] \cup ((A^m S] \cap (SA^n])$ is a subsemigroup of S. We now consider:

$$\Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big)\Big)^m S$$

$$\subseteq \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup (A^m S]\Big)^m S$$

$$= \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup (A^m S]\Big)^{m-1} \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup (A^m S]\Big) S$$

$$\subseteq \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup (A^m S]\Big)^{m-1} (AS]$$

$$= \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup (A^m S]\Big)^{m-2} \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup (A^m S]\Big) (AS]$$

$$\subseteq \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup (A^m S]\Big)^{m-2} (A^2 S]$$

$$\vdots$$

$$\subseteq (A^m S].$$

Similarly,

$$S\Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big)\Big)^n \subseteq (SA^n].$$

Then,

$$(Q^m S] \cap (SQ^n] \subseteq (A^m S] \cap (SA^n]$$

$$\subseteq \Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big),$$

where $Q = (\bigcup_{i=1}^{\max\{m+n\}} A^i] \cup ((A^m S] \cap (SA^n])$. Now,

$$\Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big] \cup \big((A^m S] \cap (SA^n]\big)\Big]$$

$$= \Big(\Big(\bigcup_{i=1}^{\max\{m,n\}} A^i\Big]\Big] \cup \Big(\big((A^m S] \cap (SA^n]\big)\Big]$$

$$\subseteq \big(\bigcup_{i=1}^{\max\{m,n\}} A^i\big] \cup \big(\big(A^m S\big] \cap \big(SA^n\big]\big).$$

Thus $\big(\bigcup_{i=1}^{\max\{m+n\}} A^i\big] \cup \big(\big(A^m S\big] \cap \big(SA^n\big]\big)$ is an (*m, n*)-quasi-ideal containing *A* of *S*, and

$$[A]_{q,(m,n)} \subseteq \big(\bigcup_{i=1}^{\max\{m+n\}} A^i\big] \cup \big(\big(A^m S\big] \cap \big(SA^n\big]\big).$$

By

$$\big(\bigcup_{i=1}^{\max\{m,n\}} A^i\big] \subseteq \big([A]_{q,(m,n)} \cup \ldots \cup [A]_{q,(m,n)}^{\max\{m,n\}}\big]$$

$$\subseteq [A]_{q,(m,n)}$$

and

$$\big(A^m S\big] \cap \big(SA^n\big] \subseteq \big(([A]_{q,(m,n)})^m S\big]$$

$$\cap \big(S([A]_{q,(m,n)})^n\big] \subseteq [A]_{q,(m,n)},$$

it follows that

$$\big(\bigcup_{i=1}^{\max\{m,n\}} A^i\big] \cup \big(\big(A^m S\big] \cap \big(SA^n\big]\big) \subseteq [A]_{q,(m,n)}.$$

This shows that (1.2) holds, and the proof is completed. □

For an element *a* of an ordered semigroup $(S,\cdot,\leq)$, we write $[\{a\}]_{q,(m,n)}$ (or simply $[a]_{q,(m,n)}$) by

$$[a]_{q,(m,n)} = \big(\bigcup_{i=1}^{\max\{m,n\}} \{a\}^i\big] \cup \big(\big(a^m S\big] \cap \big(Sa^n\big]\big).$$

In closing this section we quote the following two results proved by Sanborisoot *et al.* (2012).

Lemma 1.1. The following conditions hold for an ordered semigroup $(S,\cdot,\leq)$ and $a \in S$:

(1) $([a]_{(m,0)})^m S \subseteq (a^m S]$ for any positive integer *m*.

(2) $S([a]_{(0,n)})^n \subseteq (Sa^n]$ for any positive integer *n*.

(3) $([a]_{(m,n)})^m S([a]_{(m,n)})^n \subseteq (a^m S a^n]$ for any positive integers *m, n*.

Theorem 1.3. Let $(S,\cdot,\leq)$ be an ordered semigroup. Let *m, n* be positive integers. Let $\mathcal{R}_{(m,0)}$ be the set of all $(m,0)$-ideals of *S*, and let $\mathcal{L}_{(0,n)}$ be the set of all (0, *n*)-ideals of *S*. Then the following conditions hold:

(1) *S* is $(m,0)$-regular if and only if for all $R \in \mathcal{R}_{(m,0)}$, $R = (R^m S]$.

(2) *S* is (0, *n*)-regular if and only if for all $L \in \mathcal{L}_{(0,n)}$, $L = (SL^n]$.

2. Main Results

Let *A* be a subsemigroup of an ordered semigroup $(S,\cdot,\leq)$. For a non-empty subset *B* of *A*, we let

$$(B]_A = \{y \in A \mid y \leq b \text{ for some } b \in B\}.$$

It is clear that $(B]_A \subseteq (B]$, and the equality holds in the following lemma.

Lemma 2.1. If *A* is an (*m, n*)-ideal of an ordered semigroup $(S,\cdot,\leq)$, then $(B]_A = (B]$ for any non-empty subset *B* of *A*.

Lemma 2.2. Let *A* be an (*m, n*)-ideal of an ordered semigroup $(S,\cdot,\leq)$, and let $\varnothing \neq B \subseteq A$. Then

$$(([B_A]_{(m,n)})^m S([B_A]_{(m,n)})^n] = (B^m S B^n]$$

where $[B_A]_{(m,n)} = \big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big]_A$.

Proof. We have

$$\big(\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big]_A^m S\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big]_A^n\big]$$

$$\subseteq \big(\big(\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big)^m S\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big)^n\big]_A\big]$$

$$\subseteq \big(\big(\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big)^m S\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big)^n\big]\big]$$

$$= \big(\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big)^m S\big(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\big)^n\big].$$

Let $x \in (([B_A]_{(m,n)})^m S([B_A]_{(m,n)})^n]$. Then $x \leq y^m s z^n$ for some $s \in S$ and $y, z \in \bigcup_{i=1}^{m+n} B^i \cup B^m A B^n$. If $y, z \in \bigcup_{i=1}^{m+n} B^i$, then $y \in B^p$, $z \in B^q$ for some $p, q \in \{1, 2, \ldots, m+n\}$; hence $x \in (B^{mp} S B^{nq}] \subseteq (B^m S B^n]$. If $y \in \bigcup_{i=1}^{m+n} B^i$, $z \in B^m S B^n$, then $y \in B^p$ for some $p \in \{1, 2, \ldots, m+n\}$; hence $x \in (B^{mp} S (B^m S B^n)^n] \subseteq (B^m S B^n]$. If $y \in B^m S B^n$, $z \in \bigcup_{i=1}^{m+n} B^i$, then $z \in B^q$ for some $q \in \{1, 2, \ldots, m+n\}$; hence $x \in ((B^m S B^n)^m S B^{nq}] \subseteq (B^m S B^n]$. Finally, if $y, z \in B^m S B^n$, then $x \in ((B^m S B^n)^m S (B^m S B^n)^n] \subseteq (B^m S B^n]$. This shows that

$$\big([B_A]_{(m,n)}\big)^m S\big([B_A]_{(m,n)}\big)^n \subseteq (B^m S B^n].$$

By

$$(B^m S B^n] \subseteq (([B_A]_{(m,n)})^m S([B_A]_{(m,n)})^n],$$

it follows that

$$(([B_A]_{(m,n)})^m S([B_A]_{(m,n)})^n] = (B^m S B^n],$$

as required. This completes the proof. □

Theorem 2.1. Let A be an (m, n)-ideal of an ordered semigroup $(S,\cdot,\leq)$. Then every (m, n)-ideal of A is an (m, n)-ideal of S if and only if for each non-empty subset B of A,

$$B^m S B^n \subseteq [B_A]_{(m,n)} \tag{2.1}$$

where $[B_A]_{(m,n)} = \left(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\right]_A$.

Proof. Assume first that every (m, n)-ideal of A is an (m, n)-ideal of S. Let $\varnothing \neq B \subseteq A$. Since $[B_A]_{(m,n)}$ is an (m, n)-ideal of A, it follows by assumption that $[B_A]_{(m,n)}$ is an (m, n)-ideal of S. By Lemma 2.2,

$$B^m S B^n \subseteq (B^m S B^n] = (([B_A]_{(m,n)})^m S([B_A]_{(m,n)})^n]$$

$$\subseteq ([B_A]_{(m,n)}] = [B_A]_{(m,n)}.$$

Conversely, we assume that the equation (2.1) holds for any non-empty subset of A. Let C be an (m, n)-ideal of A. Then $C \subseteq A$ and

$$C^m S C^n \subseteq (C \cup C^2 \cup \ldots \cup C^{m+n} \cup C^m A C^n]_A \subseteq (C]_A = C.$$

By Lemma 2.1, $(C] = C$. Therefore, C is an (m, n)-ideal of S. □

For $m = 0$, $n = 1$ (resp. $m = 1$, $n = 0$), we have the following corollary:

Corollary 2.1. Let A be a left (resp. right) ideal of an ordered semigroup $(S,\cdot,\leq)$. Then every left (resp. right) ideal of A is a left (resp. right) ideal of S if and only if for each non-empty subset B of A,

$$SB \subseteq (B \cup AB]_A (resp., BS \subseteq (B \cup BA]_A).$$

Moreover we have the following, taking $m = 1$, $n = 1$:

Corollary 2.2. Let A be a bi-ideal of an ordered semigroup $(S,\cdot,\leq)$. Then every bi-ideal of A is a bi-ideal of S if and only if for each non-empty subset B of A,

$$BSB \subseteq (B \cup B^2 \cup BAB]_A.$$

Example 2.1. Let $S = \{a,b,c,d\}$ be an ordered semigroup with the multiplication and the order relation defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq = \{(a,a),(a,b),(b,b),(c,c),(d,d)\}.$$

We give the covering relation and the figure of S by:

$$\prec = \{(a,b)\}$$

Then $A = \{a,d\}$ is a bi-ideal of S, and $\{a\}$ is a bi-ideal of A. It is easy to verify that, for each non-empty subset B of A, we have $BSB \subseteq (B \cup B^2 \cup BAB]_A$. Thus, by Corollary 2.2, $\{a\}$ is a bi-ideal of S.

Theorem 2.2. Let Q be an (m, n)-quasi-ideal of an ordered semigroup $(S,\cdot,\leq)$. Then every (m, n)-quasi-ideal of Q is an (m, n)-quasi-ideal of S if and only if for each non-empty subset D of Q,

$$(D^m S] \cap (SD^n] \subseteq [D_Q]_{q,(m,n)} \tag{2.2}$$

where $[D_Q]_{q,(m,n)} = \left(\bigcup_{i=1}^{\max\{m,n\}} D^i\right]_Q \cup \left(\left(D^m Q\right]_Q \cap \left(QD^n\right]_Q\right)$.

Proof. Assume that every (m, n)-quasi-ideal of Q is an (m, n)-quasi-ideal of S. If $D \subseteq Q$ is non-empty, then, by Theorem 1.2, $[D_Q]_{q,(m,n)}$ is an (m, n)-quasi-ideal of Q. By assumption,

$$(D^m S] \cap (SD^n] \subseteq (([D_Q]_{q,(m,n)})^m S]$$

$$\cap (S([D_Q]_{q,(m,n)})^n] \subseteq [D_Q]_{q,(m,n)}.$$

Conversely, we assume that the equation (2.2) holds for any non-empty subset of Q. Let C be an (m, n)-quasi-ideal of Q. Then $C \subseteq Q$ and

$$(C^m S] \cap (SC^n] \subseteq [C_Q]_{q,(m,n)} = C.$$

By Lemma 2.1, $(C] = C$. Therefore, C is an (m, n)-quasi-ideal of S. □

For $m = 1$, $n = 1$, we have the following corollary:

Corollary 2.3. Let Q be a quasi-ideal of an ordered semigroup $(S,\cdot,\leq)$. Then every quasi-ideal of Q is a quasi-ideal of S if and only if for each non-empty subset D of Q,

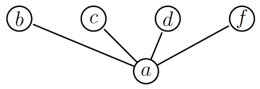
$$(DS] \cap (SD] \subseteq (D]_Q \cup ((DQ]_Q \cap (QD]_Q).$$

Example 2.2. Let $S = \{a,b,c,d,f\}$ be an ordered semigroup with the multiplication and the order relation defined by:

$\cdot$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

$\leq = \{(a,a),(a,b),(a,c),(a,d),(a,f),(b,b),(c,c),(d,d),(f,f)\}$.
We give the covering relation and the figure of *S* by:

$$\prec = \{(a,b),(a,c),(a,d),(a,f)\}$$



Then $Q = \{a,c,f\}$ is a quasi-ideal of *S*, and the quasi-ideals of *Q* are $D_1 = \{a\}$, $D_2 = \{a, c\}$ and $D_3 = \{a, f\}$. For each non-empty subset *C* of *Q*, we have $(CS] \cap (SC] \subseteq (C]_Q \cup ((CQ]_Q \cap (QC]_Q)$. By Corollary 2.3, D_1, D_2, D_3 are quasi-ideals of *S*.

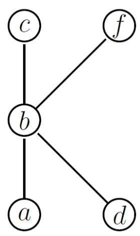
Example 2.3. Let $S = \{a,b,c,d,f\}$ be an ordered semigroup with the multiplication and the order relation defined by:

$\cdot$	a	b	c	d	f
a	d	b	b	d	f
b	b	b	b	b	f
c	b	b	c	b	f
d	d	b	b	d	f
f	b	b	f	b	f

$$\leq = \{(a,a),(a,b),(a,c),(a,f),(b,b),(b,c),(b,f),(c,c),(d,d),(d,b),(d,c),(d,f),(f,f)\}.$$

We give the covering relation and the figure of *S* by:

$$\prec = \{(a,b),(b,c),(b,f),(d,b)\}$$



It is easy to verify that $Q = \{a,b,d\}$ is an (*m, n*)-quasi-ideal of *S* for any integers $m,n \geq 2$, and the (*m, n*)-quasi-ideal of *Q* is $\{b,d\}$. For each non-empty subset *C* of *Q*, we have $(C^m S] \cap (SC^n] \subseteq [C_Q]_{q,(m,n)}$. By Theorem 2.2, $\{b,d\}$ is also a quasi-ideal of *S*.

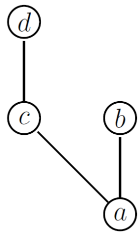
Let $(S,\cdot,\leq)$ be an ordered semigroup, and let *m, n* be non-negative integers. Then *S* is said to be (*m, n*)-*regular* (Sanborisoot *et al.*, 2012), if for any *a* in *S* there exists *x* in *S* such that $a \leq a^m x a^n$, that is, if $a \in (a^m S a^n]$.

Example 2.4. Let $S = \{a,b,c,d\}$ be an ordered semigroup with the multiplication and the order relation defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	c	c	c	c
d	c	d	c	d

$\leq = \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(c,d),(d,d)\}$.
We give the covering relation and the figure of *S* by:

$$\prec = \{(a,b),(a,c),(c,d)\}$$



Then *S* is (*m, n*)-regular for any integer $m,n \geq 1$.

Theorem 2.3. Let $(S,\cdot,\leq)$ be an ordered semigroup. Then *S* is (*m, n*)-regular if and only if

$$\forall R \in \mathcal{R}_{(m,0)}, \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (R^m L^n] \qquad (2.3)$$

where $\mathcal{R}_{(m,0)}$ is the set of all (*m*, 0)-ideals of *S* and $\mathcal{L}_{(0,n)}$ is the set of all (0, *n*)-ideals of *S*.

Proof. The assertion is obvious if $m = 0, n = 0$. If $m = 0$, $n \neq 0$, we have to show that *S* is (0, *n*)-regular if and only if $\forall L \in \mathcal{L}_{(0,n)}, L = (SL^n]$, and this follows by Theorem 1.3 (2). Similarly, for m $\neq$ 0, n = 0. This is obtained by Theorem 1.3 (1).

Finally, we let $m \neq 0, n \neq 0$. Assume that S is (*m, n*)-regular. Let $R \in \mathcal{R}_{(m,0)}$ and $L \in \mathcal{L}_{(0,n)}$. We have $(R^m L^n] \subseteq R \cap L$. Let $a \in R \cap L$. Since *S* is regular, there exists *x* in S such that $a \le a^m x a^n$. We have

$$\begin{aligned} a &\le a^m x a^n \\ &\le a^{2m-1} x a^n x a^n \\ &\le a^{3m-2} x a^n x a^n x a^n \\ &\vdots \\ &\le a^{nm-(n-1)} (x a^n)^n \\ &\in R^{nm-(n-1)} L^n \\ &\subseteq R^m L^n \\ &\subseteq (R^m L^n]. \end{aligned}$$

Thus $R \cap L \subseteq (R^m L^n]$.

Conversely, we assume that (2.3) holds. Let $a \in S$. Since $[a]_{(m,0)} \in \mathcal{R}_{(m,0)}$ and $S \in \mathcal{L}_{(0,n)}$, we have

$$[a]_{(m,0)} = [a]_{(m,0)} \cap S = (([a]_{(m,0)})^m S^n] \subseteq (([a]_{(m,0)})^m S].$$

By Lemma 1.1, $[a]_{(m,0)} \subseteq (a^m S]$. Similarly, $[a]_{(0,n)} \subseteq (Sa^n]$. From

$$\begin{aligned} a &\in [a]_{(m,0)} \cap [a]_{(0,n)} \\ &\subseteq (a^m S] \cap (Sa^n] \\ &= ((a^m S])^m ((Sa^n])^n \\ &\subseteq (a^m S](Sa^n] \\ &\subseteq (a^m S a^n], \end{aligned}$$

we conclude that S is (m, n)-regular. We now complete the proof.

Corollary 2.4. Let $(S, \cdot, \le)$ be an ordered semigroup. Then S is (m, n)-regular if and only if

$$\forall a \in S, [a]_{(m,0)} \cap [a]_{(0,n)} = (([a]_{(m,0)})^m ([a]_{(0,n)})^n].$$

Theorem 2.4. Let $(S, \cdot, \le)$ be an ordered semigroup. Then S is (m, n)-regular if and only if

$$\forall a \in S, [a]_{(m,n)} = (a^m S a^n].$$

Proof. Assume that S is (m, n)-regular. Let $a \in S$ and $x \in [a]_{(m,n)}$. Then, by Theorem 1.1, $x \le y$ for some y in $\bigcup_{i=1}^{m+n} a^i \cup a^m S a^n$. If $y \in a^m S a^n$, we are done. Suppose that $y \in \bigcup_{i=1}^{m+n} a^i$; then $y = a^p$ for some $p \in \{1, 2, \ldots, m+n\}$. We have

$$x \in (a^p] \subseteq ((a^m S a^n]^p] \subseteq ((a^m S a^n]] = (a^m S a^n].$$

Since $(a^m S a^n] \subseteq [a]_{(m,n)}, [a]_{(m,n)} = (a^m S a^n]$.

Conversely, if $a \in S$, then $a \in [a]_{(m,n)} = (a^m S a^n]$, and hence S is (m, n)-regular.

Example 2.5. We consider the ordered semigroup which is defined in Example 2.2. We have $[a]_{(1,1)} = (a]$, $[b]_{(1,1)} = (\{a, b\}]$, $[c]_{(1,1)} = (\{a, c\}]$, $[d]_{(1,1)} = (\{a, d\}]$, and $[f]_{(1,1)} = (\{a, f\}]$. Then, by Theorem 2.4, S is regular.

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