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Original Article

Spectrum and fine spectrum of the lower triangular matrix $B(r,s,t)$ over the sequence space cs

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Abstract

Fine spectra of various matrix operators on different sequence spaces have been examined by several authors. Recently, some authors have determined the approximate point spectrum, the defect spectrum and the compression spectrum of various matrix operators on different sequence spaces. Here in this article we have determined the spectrum and fine spectrum of the lower triangular matrix $B(r,s,t)$ on the sequence space *cs*. In a further development, we have also determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $B(r,s,t)$ on the sequence space *cs*.

Keywords: spectrum of an operator, matrix mapping, sequence space

1. Introduction

By *w*, we denote the space of all real or complex valued sequences. Throughout the article c, c_0, bv, bs, ℓ_1, ℓ_∞ represent the spaces of all convergent, null, bounded variation, bounded series, absolutely summable and bounded sequences respectively. Also bv_0 denotes the sequence space $bv \cap c_0$.

Fine spectra of various matrix operators on different sequence spaces have been examined by several authors. The spectrum and fine spectrum of the Zweier Matrix on the sequence space ℓ_1 and bv was studied by Altay and Karakuş (2005). Altay and Başar (2004, 2005) determined the fine spectrum of the difference operator Δ and the generalized difference operator $B(r,s)$ on the sequence spaces c_0 and c. Furkan *et al.* (2006) have determined the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv. Altun (2011a, 2011b) determined the fine spectrum of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. The fine spectra of the Cesàro operator C_1 over the sequence space bv_p, $(1 \le p < \infty)$ was determined by Akhmedov and Başar (2008). Okutoyi (1990) determined the spectrum of the Cesàro operator C_1 on the sequence space bv_0. Fine spectra of operator $B(r,s,t)$ over the sequence spaces ℓ_1 and bv and generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p, $(1 \le p < \infty)$, were studied by Bilgiç and Furkan (2007, 2008). Fine spectrum of the generalized difference operator Δ_v on the sequence space ℓ_1 was investigated by Srivastava and Kumar (2010). Panigrahi and Srivastava (2011, 2012) studied the spectrum and fine spectrum of the second order difference operator Δ^2_{uv} on the sequence space c_0 and generalized second order forward difference operator Δ^2_{uvw} on the sequence space ℓ_1. Fine spectra of upper triangular double-band matrices $U(r,s)$ over the sequence spaces c_0 and c was studied by Karakaya and Altun (2010). Karaisa and Başar (2013) have determined the spectrum and fine spectrum of the upper triangular matrix $A(r,s,t)$ over the sequence space ℓ_p $(0 < p < \infty)$. In a further development, they have also

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determined the approximate point spectrum, defect spectrum and compression spectrum of the operator $A(r,s,t)$ on the sequence space $\ell_p\left(0 < p < \infty\right)$.The approximate point spectrum, defect spectrum and compression spectrum of the operator $B(r,s)$ on the sequence spaces c_0, c, ℓ_p and bv_p, $\left(1 < p < \infty\right)$ were studied by Başar, Durna and Yildirim (2011).

The notion of matrix transformations over sequence space has been studied from various aspects. Besides the above listed workers, the spectrum and fine spectrum for various matrix operators has been investigated by Tripathy and Das (2014, 2015), Tripathy and Pal (2013a, 2013b, 2014), Tripathy and Saikia (2013) and many others in recent years.

In this paper, we will determine the spectrum and fine spectrum of the lower triangular matrix $B(r,s,t)$ on the sequence space cs. Also, we will determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $B(r,s,t)$ on the sequence space cs. Clearly, $cs = \left\{x = \left(x_n\right) \in w : \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n} x_i \text{ exists}\right\}$ is a Banach space with respect to the norm $\|x\|_{cs} = \sup\limits_{n} \left|\sum\limits_{i=0}^{n} x_i\right|$.

2. Preliminaries and Background

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R\left(T\right)$, we denote the range of T, i.e.

$$R\left(T\right) = \left\{y \in Y : y = Tx, x \in X\right\}.$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B\left(X\right)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $\left(T^* f\right)\left(x\right) = f\left(Tx\right)$, for all $f \in X^*$ and $x \in X$.

Let $X \neq \left\{\theta\right\}$ be a complex normed linear space and $T : D\left(T\right) \to X$ be a linear operator with domain $D\left(T\right) \subseteq X$. With T, we associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = \left(T - \lambda I\right)^{-1},$$

and call it the *resolvent* operator of T.

Let $X \neq \left\{\theta\right\}$ be a complex normed linear space and $T : D\left(T\right) \to X$ be a linear operator with domain $D\left(T\right) \subseteq X$. A *regular value* λ of T is a complex number such that

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X i.e. $\overline{R\left(T_\lambda\right)} = X$.

The *resolvent set* of T, denoted by $\rho\left(T, X\right)$, is the set of all regular values λ of T. Its complement $\sigma\left(T, X\right) = C \setminus \rho\left(T, X\right)$ in the complex plane C is called the *spectrum* of T. Furthermore, the spectrum $\sigma\left(T, X\right)$ is partitioned into three disjoint sets as follows:

The *point* (*discrete*) *spectrum* $\sigma_p\left(T, X\right)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p\left(T, X\right)$ is called an eigenvalue of T.

The *continuous spectrum* $\sigma_c\left(T, X\right)$ is the set such that T_λ^{-1} exists and satisfies (R3), but not (R2), that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r\left(T, X\right)$ is the set such that T_λ^{-1} exists (and may be bounded or not), but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.

If X is a Banach space and $T \in B\left(X\right)$, then there are three possibilities for $R(T)$ and T^{-1}:

(I) $R\left(T\right) = X$,

(II) $R\left(T\right) \neq \overline{R\left(T\right)} = X$,

(III) $\overline{R\left(T\right)} \neq X$,

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

(One may refer to Goldberg (1985))

Applying Goldberg's classification to T_λ, we have three possibilities for T_λ and T_λ^{-1};

(I) T_λ is surjective,

(II) $R\left(T_\lambda\right) \neq \overline{R\left(T_\lambda\right)} = X$,

(III) $\overline{R\left(T_\lambda\right)} \neq X$,

and

(1) T_λ is injective and T_λ^{-1} is continuous,

(2) T_λ is injective but T_λ^{-1} is discontinuous,

(3) T_λ is not injective.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in Table 1.

These are labeled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2 and . If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in II_1$, then λ is in the resolvent set $\rho\left(T, X\right)$ of T. The further classification gives rise to the fine spectrum of T. If an operator is in state II_2 for example, then $R\left(T\right) \neq \overline{R\left(T\right)} = X$ and T^{-1} exists but is discontinuous and we write $\lambda \in II_2\sigma\left(T, X\right)$.

Again, following Appell *et al.* (2004), we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Table 1. Subdivisions of spectrum of a linear operator

	I	II	III
1	$\rho\left(T, X\right)$		$\sigma_r\left(T, X\right)$
2	$\sigma_c\left(T, X\right)$	$\sigma_c\left(T, X\right)$	$\sigma_r\left(T, X\right)$
3	$\sigma_p\left(T, X\right)$	$\sigma_p\left(T, X\right)$	$\sigma_p\left(T, X\right)$

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$ as $k \to \infty$.

The *approximate point spectrum of T*, denoted by $\sigma_{ap}(T, X)$, is defined as the set

$$\sigma_{ap}(T, X) = \{\lambda \in C : \textit{there exists a Weyl sequence for } T - \lambda I\} \tag{2.1}$$

The *defect spectrum* of T, denoted by $\sigma_\delta(T, X)$, is defined as the set

$$\sigma_\delta(T, X) = \{\lambda \in C : T - \lambda I \textit{ is not surjective}\} \tag{2.2}$$

The two subspectra given by (2.1) and (2.2) form a (not necessarily disjoint) subdivisions

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_\delta(T, X) \tag{2.3}$$

of the spectrum. There is another subspectrum, $\sigma_{co}(T, X) = \{\lambda \in C : \overline{R(T - \lambda I)} \neq X\}$ which is often called the *compression spectrum of T*. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X) \tag{2.4}$$

Clearly, $\sigma_p(T, X) \subseteq \sigma_{ap}(T, X)$ and $\sigma_{co}(T, X) \subseteq \sigma_\delta(T, X)$. Moreover, it is easy to verify that

$$\sigma_r(T, X) = \sigma_{co}(T, X) \setminus \sigma_p(T, X) \quad \text{and}$$

$$\sigma_c(T, X) = \sigma(T, X) \setminus [\sigma_p(T, X) \cup \sigma_{co}(T, X)]$$

By the definitions given above, we can illustrate the subdivisions spectrum in Table 2.

Proposition 2.1. [Appell *et al.* (2004), Proposition 1.3, p.28]: Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:

(a) $\sigma(T^*, X^*) = \sigma(T, X)$.

(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.

(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$.

(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$.

(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.

(f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.

(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*)$
$= \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal.

Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (Appell *et al.*, 2004).

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N_0 = \{0, 1, 2, \ldots\ldots\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n \in N_0, \tag{2.5}$$

Table 2. Subdivisions of spectrum of a linear operator

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is bounded
I	$R(T - \lambda I) = X$	$\lambda \in \rho(T, X)$	---	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$
II	$\overline{R(T - \lambda I)} = X$	$\lambda \in \rho(T, X)$	$\lambda \in \sigma_c(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$
III	$\overline{R(T - \lambda I)} \neq X$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$

By $(\lambda : \mu)$, we denote the class of all matrices such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right hand side of (2.5) converges for each $n \in N_0$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in N_0} \in \mu$ for all $x \in \lambda$.

The lower triangular matrix $B(r,s,t)$ is an infinite matrix of the form

$$B(r,s,t) = \begin{bmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ t & s & r & 0 & \cdots \\ 0 & t & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We assume here and hereafter that s and t are complex parameters which do not simultaneously vanish.

The following results will be used in order to establish the results of this article.

Lemma 2.2 [Wilansky (1984), Example 6B, Page 130]. *The matrix* $A = (a_{nk})$ *gives rise to a bounded linear operator* $T \in B(cs)$ *from cs to itself if and only if*

(i) $\sup_m \sum_k \left| \sum_{n=1}^{m} \left| a_{nk} - a_{n,k-1} \right| \right| < \infty$

(ii) $\sum_n a_{nk}$ *is convergent for each k.*

Lemma 2.3 [Golberg (1985), Page 59] *T has a dense range if and only if* T^* *is one to one.*

Lemma 2.4 [Golberg (1985), Page 60] *T has a bounded inverse if and only if* T^* *is onto.*

3. Spectrum and fine spectrum of the operator *B*(*r,s,t*) on the sequence space *cs*

In this section, the fine spectrum of the operator $B(r,s,t)$ on the sequence space has been examined.

Before giving the main theorem we should give the following remark. In this work, here and in follows, if z is a complex number then by $\sqrt{z}$ we always mean the square root of with non-negative real part. If $\operatorname{Re}\left(\sqrt{z}\right) = 0$ then $\sqrt{z}$ represents square root of z with $\operatorname{Im}\left(\sqrt{z}\right) \geq 0$. The same results are obtained if $\sqrt{z}$ represents the square root.

Theorem 3.1 $B(r,s,t) : cs \to cs$ *is a bounded linear operator and* $\left\| B(r,s,t) \right\|_{(cs\,:\,cs)} \leq |r| + |s| + |t|$.

Proof: From Lemma 2.2, it is easy to show that $B(r,s,t)$: $cs \to cs$ is a bounded linear operator.

Now,

$$\begin{aligned} \left| B(r,s,t)(x) \right| &= \left| \sum_{i=0}^{n} r x_i + \sum_{i=0}^{n-1} s x_i + \sum_{i=0}^{n-2} t x_i \right| \\ &\leq |r| \left| \sum_{i=0}^{n} x_i \right| + |s| \left| \sum_{i=0}^{n-1} x_i \right| + |t| \left| \sum_{i=0}^{n-2} x_i \right| \\ &\leq \left(|r| + |s| + |t| \right) \|x\|_{cs} \end{aligned}$$

and hence, $\left\| B(r,s,t) \right\|_{(cs\,:\,cs)} \leq |r| + |s| + |t|$.

Theorem 3.2 *If s is a complex number such that* $\sqrt{s^2} = -s$, *then* $\sigma\left(B(r,s,t), cs\right) = S$ where

$$S = \left\{ \alpha \in C : \left| \frac{2(r-\alpha)}{-s + \sqrt{s^2 - 4t(r-\alpha)}} \right| \leq 1 \right\}.$$

Proof: We shall prove this theorem by showing that $\left(B(r,s,t) - \alpha I\right)^{-1}$ exists and is in $(cs : cs)$ for $\alpha \notin S$, and then show that the operator $B(r,s,t) - \alpha I$ is not invertible for $\alpha \in S$.

Without loss of any generality we assume that $\sqrt{s^2} = -s$. Let $\alpha \notin S$. Clearly $\alpha \neq r$ and so $B(r,s,t) - \alpha I$ is a triangle, therefore $\left(B(r,s,t) - \alpha I\right)^{-1}$ exists. Let $y = (y_n) \in cs$. Solving $\left(B(r,s,t) - \alpha I\right) x = y$ for x in terms of y we get

$$\begin{aligned} x_0 &= \frac{y_0}{r-\alpha} \\ x_1 &= \frac{y_1}{r-\alpha} + \frac{-s y_0}{(r-\alpha)^2} \\ x_2 &= \frac{y_{21}}{r-\alpha} + \frac{-s y_1}{(r-\alpha)^2} + \frac{\left[s^2 - t(r-\alpha) \right] y_0}{(r-\alpha)^3} \\ &\;\;\vdots \end{aligned}$$

Let us denote $a_1 = \dfrac{1}{r-\alpha}$, $a_2 = \dfrac{-s}{(r-\alpha)^2}$, $a_3 = \dfrac{s^2 - t(r-\alpha)}{(r-\alpha)^3}$ etc.

Then, we have

$$\begin{aligned} x_0 &= a_1 y_0 \\ x_1 &= a_1 y_1 + a_2 y_0 \\ x_2 &= a_1 y_2 + a_2 y_1 + a_3 y_0 \\ &\;\;\vdots \\ x_n &= a_1 y_n + a_2 y_{n-1} + a_3 y_{n-2} + \cdots + a_{n+1} y_0 = \sum_{k=0}^{n} a_{n+1-k} y_k. \end{aligned}$$

That is

$$\left(B(r,s,t)-\alpha I\right)^{-1}=\left(a_{nk}\right)=\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 0 & 0 & \cdots \\ a_4 & a_3 & a_2 & a_1 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Also, from $\left(B(r,s,t)-\alpha I\right)x=y$, we have $y_n=tx_{n-2}+sx_{n-1}+(r-\alpha)x_n$.

Using the recurrence relation $x_n=\sum_{k=0}^{n}a_{n+1-k}y_k$ we get

$$\begin{aligned} y_n &= t\sum_{k=0}^{n-2}a_{n-1-k}y_k+s\sum_{k=0}^{n-1}a_{n-k}y_k+(r-\alpha)\sum_{k=0}^{n}a_{n+1-k}y_k \\ &= y_0\left(ta_{n-1}+sa_n+(r-\alpha)a_{n+1}\right)+y_1\left(ta_{n-2}+sa_{n-1}+(r-\alpha)a_n\right) \\ &\quad +\cdots+y_na_1(r-\alpha). \end{aligned}$$

This gives

$$\begin{aligned} &\left(ta_{n-1}+sa_n+(r-\alpha)a_{n+1}\right)=0 \\ &\left(ta_{n-2}+sa_{n-1}+(r-\alpha)a_n\right)=0 \\ &\cdots \\ &a_1(r-\alpha)=1 \end{aligned}$$

This sequence can be obtained recursively by putting

$$a_1=\frac{1}{r-\alpha},\quad a_2=\frac{-s}{(r-\alpha)^2},\quad ta_{n-2}+sa_{n-1}+(r-\alpha)a_n=0,\quad n\ge 3.$$

The characteristic equation of the recurrence relation is $(r-\alpha)\lambda^2+s\lambda+t=0$.

Then we have two cases:

Case 1: Let $D=s^2-4t(r-\alpha)\neq 0$.

Then the roots of the characteristic equation are

$$\lambda_1=\frac{-s+\sqrt{D}}{2(r-\alpha)}\quad\text{and}\quad\lambda_2=\frac{-s-\sqrt{D}}{2(r-\alpha)}.$$

It is easy to show that $a_n=\dfrac{\lambda_1^n-\lambda_2^n}{\sqrt{D}}$, $n\ge 1$.

Since $\alpha\notin S$, so $|\lambda_1|<1$ and therefore we have

$$\left|1+\sqrt{\frac{D}{s^2}}\right|<\left|\frac{2(r-\alpha)}{-s}\right|.$$

Since, $|1-\sqrt{z}|\le|1+\sqrt{z}|$ for all $z\in C$, so $\left|1-\sqrt{\dfrac{D}{s^2}}\right|<\left|\dfrac{2(r-\alpha)}{-s}\right|$ and hence $|\lambda_2|<1$.

It is easy to show that for all m,

$$\sum_k\left|\sum_{n=1}^{m}\left(a_{nk}-a_{n,k-1}\right)\right|\le\sum_{n=0}^{m}|a_n|=\frac{1}{\sqrt{D}}\left(\sum_{n=0}^{m}|\lambda_1|^n+\sum_{n=0}^{m}|\lambda_2|^n\right) \tag{3.1}$$

and hence, $\sup_m\sum_k\left|\sum_{n=1}^{m}(a_{nk}-a_{n,k-1})\right|<\infty$, as $|\lambda_1|<1$ and $|\lambda_2|<1$.

Since $|\lambda_1|<1$ and $|\lambda_2|<1$, so for all k, the series

$$\sum_n a_{nk}=a_1+a_2+a_3+\cdots \tag{3.2}$$

is absolutely convergent and hence convergent.

So, by Lemma 2.2, $\left(B(r,s,t)-\alpha I\right)^{-1}$ is in $(cs:cs)$.

This shows that $\sigma\left(B(r,s,t),cs\right)\subseteq S$.

Next let $\alpha\in S$. If $\alpha=r$ then $B(r,s,t)-\alpha I$ is represented by the matrix

$$B(0,s,t)=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ s & 0 & 0 & 0 & \cdots \\ t & s & 0 & 0 & \cdots \\ 0 & t & s & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $B(r,s,t)-rI=B(0,s,t)$ does not have a dense range, it is not invertible.

So we may assume that $\alpha\neq r$. Since $D=s^2-4t(r-\alpha)\neq 0$ therefore we must have $|\lambda_1|>|\lambda_2|$, from which we have $\lim_{n\to\infty}a_n\neq 0$ and so for all k, the series

$$\sum_n a_{nk}=a_1+a_2+a_3+\cdots$$

is divergent. Therefore $\left(B(r,s,t)-\alpha I\right)^{-1}$ is not in $(cs:cs)$ and hence $S\subseteq\sigma\left(B(r,s,t),cs\right)$.

Case 2: Let $D=s^2-4t(r-\alpha)=0$. Then for all $n\ge 1$, we get

$$a_n=\left(\frac{2n}{-s}\right)\left(\frac{-s}{2(r-\alpha)}\right)^n$$

Since $\alpha\notin S$, so $\left|\dfrac{-s}{2(r-\alpha)}\right|<1$.

Then for all m,

$$\sum_k\left|\sum_{n=1}^{m}\left(a_{nk}-a_{n,k-1}\right)\right|\le\sum_{n=0}^{m}|a_n|\le\sum_{n=0}^{\infty}|a_n|$$

and hence, $\sup_m\sum_k\left|\sum_{n=1}^{m}(a_{nk}-a_{n,k-1})\right|<\infty$, as $\left|\dfrac{-s}{2(r-\alpha)}\right|<1$.

Since $\left|\frac{-s}{2(r-\alpha)}\right| < 1$, so for all *k*, the series

$$\sum_n a_{nk} = a_1 + a_2 + a_3 + \cdots$$

is absolutely convergent and hence convergent. So, by Lemma 2.2, $\left(B(r,s,t)-\alpha I\right)^{-1}$ is in $(cs:cs)$. This shows that $\sigma\left(B(r,s,t),cs\right) \subseteq S$.

Next let $\alpha \in S$. Then we have $\left|\frac{-s}{2(r-\alpha)}\right| \geq 1$ from which we get $\lim_{n\to\infty} a_n \neq 0$ and so for all *k*, the series

$$\sum_n a_{nk} = a_1 + a_2 + a_3 + \cdots$$

is divergent. Therefore $\left(B(r,s,t)-\alpha I\right)^{-1}$ is not in $(cs:cs)$ and hence $S \subseteq \sigma\left(B(r,s,t),cs\right)$.

Thus in each case we get $\sigma\left(B(r,s,t),cs\right) = S$. This completes the proof.

Remark: If $\sqrt{s^2} = s$, then we obtain the same sequence and

$$\sigma\left(B(r,s,t),cs\right) = \left\{\alpha \in C : \left|\frac{2(r-\alpha)}{s+\sqrt{s^2-4t(r-\alpha)}}\right| \leq 1\right\}$$

.

Theorem 3.3 *The point spectrum of the operator* $B(r,s,t)$ *over is given by* $\sigma_p\left(B(r,s,t),cs\right) = \varnothing$.

Proof: Let α be an eigenvalue of the operator $B(r,s,t)$. Then there exists $x \neq \theta = (0,0,0,...)$ in *cs* such that $B(r,s,t)x = \alpha x$.
Then, we have

$$\begin{aligned}
&rx_0 = \alpha x_0\\
&sx_0 + rx_1 = \alpha x_1\\
&tx_0 + sx_1 + rx_2 = \alpha x_2\\
&tx_1 + sx_2 + rx_3 = \alpha x_3\\
&\ldots\\
&tx_{n-2} + sx_{n-1} + rx_n = \alpha x_n, \qquad n \geq 2
\end{aligned}$$

If x_k is the first non-zero entry of the sequence (x_n), then $\alpha = r$. Then from the relation $tx_{k-1} + sx_k + rx_{k+1} = \alpha x_{k+1}$, we have $x_k = 0$, a contradiction.
Hence, $\sigma_p\left(B(r,s,t),cs\right) = \varnothing$. This completes the proof. □

If $T: cs \to cs$ is a bounded linear operator represented by a matrix *A*, then it is known that the adjoint operator $T^*: cs^* \to cs^*$ is defined by the transpose A' of the matrix *A*.

It should be noted that the dual space cs^* of *cs* is isometrically isomorphic to the Banach space *bv* of all bounded variation sequences normed by $\|x\|_{bv} = \sum_{n=0}^{\infty}|x_{n+1} - x_n| + \lim_{n\to\infty}|x_n|$.

Theorem 3.4 *The point spectrum of the operator* $B(r,s,t)^*$ *over* cs^* *is given by* $\sigma_p\left(B(r,s,t)^*, cs^* \cong bv\right) = S_1$, where

$$S_1 = \left\{\alpha \in C : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1\right\}.$$

Proof: Let α be an eigenvalue of the operator $B(r,s,t)^*$. Then there exists $x \neq \theta = (0,0,0,\cdots)$ in *bv* such that $B(r,s,t)^*$ $x = ax$. Then, we have

$$\begin{aligned}
&B(r,s,t)^t x = \alpha x\\
\Rightarrow\ &rx_0 + sx_1 + tx_2 = \alpha x_0\\
&rx_1 + sx_2 + tx_3 = \alpha x_1\\
&rx_2 + sx_3 + tx_4 = \alpha x_2\\
&\ldots\\
&rx_n + sx_{n+1} + tx_{n+2} = \alpha x_n\ ,\ \ n \geq 0.
\end{aligned}$$

It is clear that if $\alpha = r$ then we may choose $x_0 \neq 0$ and $x = (x_0, 0, 0, 0, ...)$ is an eigenvector corresponding to $\alpha = r$. Assume that $\alpha \neq r$.
Then, we have

$$\begin{aligned}
x_2 &= \frac{-s}{t}x_1 - \frac{r-\alpha}{t}x_0\\
x_3 &= \frac{s^2 - t(r-\alpha)}{t^2}x_1 + \frac{s(r-\alpha)}{t^2}x_0\\
&\vdots\\
x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0, \qquad n \geq 2.
\end{aligned}$$

If $\alpha \in S_1$, then we may choose

$$x_0 = 1, \quad x_1 = \frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}.$$

We now show that $x_n = (x_1)^n$, $n \geq 2$.

Since λ_1 and λ_2 are roots of the characteristic equation $(r-\alpha)\lambda^2 + s\lambda + t = 0$, therefore

$$\lambda_1\lambda_2 = \frac{t}{r-\alpha}, \quad \lambda_1 - \lambda_2 = \frac{\sqrt{D}}{r-\alpha}$$

where $\lambda_1 = \frac{-s+\sqrt{D}}{2(r-\alpha)}$, $\lambda_2 = \frac{-s-\sqrt{D}}{2(r-\alpha)}$ and $D = s^2 - 4t(r-\alpha)$ $\neq 0$.

Clearly $x_1 = \frac{1}{\lambda_1}$. Then we have

$$\begin{aligned} x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0, \qquad n \ge 2 \\ &= \left(\frac{r-\alpha}{t}\right)^{n-1}(r-\alpha)\left(-a_{n-1}x_0 + a_n x_1\right) \\ &= \frac{1}{(\lambda_1\lambda_2)^{n-1}}\frac{r-\alpha}{\sqrt{D}}\left(-\lambda_1^{n-1}+\lambda_2^{n-1}+\lambda_1^{n-1}-\lambda_2^{n}\lambda_1^{-1}\right) \\ &= \frac{1}{\lambda_1^{n-1}\lambda_2^{n-1}}\left(\frac{1}{\lambda_1-\lambda_2}\right)\lambda_2^{n-1}\left(\frac{\lambda_1-\lambda_2}{\lambda_1}\right) \\ &= \frac{1}{\lambda_1^{n}} \\ &= (x_1)^n \end{aligned}$$

If $D = s^2 - 4t(r-\alpha) = 0$ then also we may get the same result.

Now, $\sum_{n=0}^{\infty}|x_{n+1}-x_n| \le \sum_{n=0}^{\infty}|x_{n+1}| + \sum_{n=0}^{\infty}|x_n| = \sum_{n=0}^{\infty}|x_1|^{n+1} + \sum_{n=0}^{\infty}|x_1|^{n} < \infty$ as $|x_1| < 1$. Therefore $x \in bv$.

Hence $S_1 \subseteq \sigma_p\left(B(r,s,t)^*, cs^* \cong bv\right)$.

Next assume that $\alpha \notin S_1$. Then $\left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| \ge 1$ and so $|\lambda_1| \le 1$. We must show that $\alpha \notin \sigma_p\left(B(r,s,t)^*, cs^* \cong bv\right)$.

Using $x_n = \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0$, $n \ge 2$, we get

$$\frac{x_{n+1}}{x_n} = \left(\frac{r-\alpha}{t}\right)\frac{a_n}{a_{n-1}}\left(\frac{-x_0 + \frac{a_{n+1}}{a_n}x_1}{-x_0 + \frac{a_n}{a_{n-1}}x_1}\right).$$

We now consider three cases:

Case (i): $|\lambda_2| < |\lambda_1| \le 1$

In this case $D = s^2 - 4t(r-\alpha) \neq 0$ and

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1^{n}-\lambda_2^{n}} = \lim_{n\to\infty}\frac{\lambda_1^{n+1}\left[1-\left(\frac{\lambda_2}{\lambda_1}\right)^{n+1}\right]}{\lambda_1^{n}\left[1-\left(\frac{\lambda_2}{\lambda_1}\right)^{n}\right]} = \lambda_1$$

Now, if $-x_0 + \lambda_1 x_1 = 0$, then we get $x_n = \frac{x_0}{\lambda_1^n}$. Since $|\lambda_1| \le 1$, therefore $(x_n) \notin c$ and so $(x_n) \notin bv$. Otherwise

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{|\lambda_1||\lambda_2|}|\lambda_1| = \frac{1}{|\lambda_2|} > 1.$$

Case (ii): $|\lambda_2| = |\lambda_1| < 1$.

In this case $D = s^2 - 4t(r-\alpha) = 0$ and $a_n = \left(\frac{2n}{-s}\right)\left(\frac{-s}{2(r-\alpha)}\right)^n$, $n \ge 1$.
Then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{-s}{2(r-\alpha)} = \lambda_1 = \lambda_2$$

and so

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{|\lambda_1||\lambda_2|}|\lambda_1| = \frac{1}{|\lambda_2|} > 1.$$

Case (iii): $|\lambda_2| = |\lambda_1| = 1$

In this case $D = s^2 - 4t(r-\alpha) = 0$ and we have $\left|\frac{-s}{2t}\right| = 1$.

Assume that $\alpha \in \sigma_p\left(B(r,s,t)^*, cs^* \cong bv\right)$. This implies that $x \in bv$ and $x \neq \theta$.

Again from $x_n = \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0$, $n \ge 2$ we get

$$x_n = \left(\frac{-s}{2t}\right)^{n-1}\left[-(n-1)\left(\frac{-s}{2t}\right)x_0 + nx_1\right]$$

Now, $x \in bv$ and so $x \in c$. Therefore we must have $x_0 = x_1 = 0$. Which implies $x = \theta$, a contradiction. So $\alpha \notin \sigma_p\left(B(r,s,t)^*, cs^* \cong bv\right)$.

In case (i) and case (ii) above, we have $(x_n) \notin c$ and so $(x_n) \notin bv$. In case (iii) by assuming $\alpha \in \sigma_p\left(B(r,s,t)^*, cs^* \cong bv\right)$ we get a contradiction.

This completes the proof.

Theorem 3.5 *The residual spectrum of the operator* $B(r,s,t)$ *over cs is given by*

$$\sigma_r\left(B(r,s,t), cs\right) = S_1.$$

Proof: Since, $\sigma_r\left(B(r,s,t),cs\right) = \sigma_p\left(B(r,s,t)^*,cs^*\right) \setminus \sigma_p\left(B(r,s,t),cs\right)$, so we get the required result by using Theorem 3.3 and Theorem 3.4 .

Theorem 3.6 *The continuous spectrum of the operator* $B(r,s,t)$ *over are given by* $\sigma_c\left(B(r,s,t),cs\right) = S_2$, where

$$S_2 = \left\{\alpha \in C : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| = 1\right\}.$$

Proof: Since, $\sigma\left(B(r,s,t),cs\right)$ is the disjoint union of $\sigma_p\left(B(r,s,t),cs\right)$, $\sigma_r\left(B(r,s,t),cs\right)$ and $\sigma_c\left(B(r,s,t),cs\right)$, therefore, by Theorem 3.3, Theorem 3.4 and Theorem 3.5, we get $\sigma_c\left(B(r,s,t),cs\right) = \left\{\alpha \in C : \left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| = 1\right\}$.□

Theorem 3.7 *If* $\alpha = r$, *then* $\alpha \in III_1\sigma\left(B(r,s,t),cs\right)$ *if* $|t| < |s|$ *and* $\alpha \in III_2\sigma\left(B(r,s,t),cs\right)$ *if* $|t| \geq |s|$.

Proof: If $\alpha = r$, the range of $B(r,s,t)$ is not dense. So, from Table 2 and Theorem 3.5, we have $\alpha \in \sigma_r\left(B(r,s,t),cs\right)$.

From Table 2, we get $\sigma_r\left(B(r,s,t),cs\right) = III_1\sigma\left(B(r,s,t),cs\right) \cup III_2\sigma\left(B(r,s,t),cs\right)$.
Therefore, $\alpha \in III_1\sigma\left(B(r,s,t),cs\right)$ or $\alpha \in III_2\sigma\left(B(r,s,t),cs\right)$.
Also for $\alpha = r$, $B(r,s,t) - \alpha I = B(0,s,t)$.
A left inverse of $B(0,s,t)$ is

$$\left(B(0,s,t)\right)^{-1} = \begin{bmatrix} 0 & \frac{1}{s} & 0 & 0 & \cdots \\ 0 & \frac{(-t)}{s^2} & \frac{1}{s} & 0 & \cdots \\ 0 & \frac{(-t)^2}{s^3} & \frac{(-t)}{s^2} & \frac{1}{s} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In other words $\left(B(0,s,t)\right)^{-1} = (b_{nk})$, where

$$b_{nk} = \begin{cases} \frac{(-t)^{n+1-k}}{s^{n+2-k}}, & \text{if } 1 < k < n+2 \\ 0, \text{ if } k = 1 \text{ or } k \geq n+2 \end{cases}$$

Now for each m, we get

$$\sum_k \left|\sum_{n=1}^m \left(b_{nk} - b_{n,k-1}\right)\right| \leq \frac{1}{|s|} + \frac{|t|}{|s|^2} + \frac{|t|^2}{|s|^3} + \cdots + \frac{|t|^{m-1}}{|s|^m}$$ and so

$\sup_m \sum_k \left|\sum_{n=1}^m \left(b_{nk} - b_{n,k-1}\right)\right|$ exists if and only if $|t| < |s|$.

Also for each k, $\sum_n b_{nk} = \frac{1}{s} + \frac{-t}{s^2} + \frac{(-t)^2}{s^3} + \cdots$ is convergent if and only if $|t| < |s|$.
Therefore, the matrix $\left(B(0,s,t)\right)^{-1}$ is in $(cs : cs)$ if $|t| < |s|$ and not in $(cs : cs)$ if $|t| \geq |s|$.
This completes the theorem.

Theorem 3.8 *If* $\alpha \neq r$ *and* $\alpha \in \sigma_r\left(B(r,s,t),cs\right)$, *then* $\alpha \in III_2\sigma\left(B(r,s,t),cs\right)$.

Proof: Since, $\alpha \in \sigma_r\left(B(r,s,t),cs\right)$, therefore, from Table 2, we have $\alpha \in III_1\sigma\left(B(r,s,t),cs\right)$ or $\alpha \in III_2\sigma\left(B(r,s,t),cs\right)$.

Now, $\alpha \in \sigma_r\left(B(r,s,t),cs\right)$ implies that $\left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1$

and so $|\lambda_1| > 1$

Therefore, the series (3.1) in Theorem 3.2 is not convergent and hence, the operator $B(r,s,t)$ has no bounded inverse. Therefore, $\alpha \in III_2\sigma\left(B(r,s,t),cs\right)$.

Theorem 3.9 *The approximate point spectrum of the operator* $B(r,s,t)$ *over is given by* $\sigma_{ap}\left(B(r,s,t),cs\right) =$

$$\begin{cases} S \setminus \{r\} \text{ , if } |t| < |s| \\ S, \quad \text{if } |t| \geq |s| \end{cases}.$$

Proof: From Table 2, we have $\sigma_{ap}\left(B(r,s,t),cs\right) = \sigma\left(B(r,s,t),cs\right) \setminus III_1\sigma\left(B(r,s,t),cs\right)$.
Using Theorem 3.2 and Theorem 3.7, we get the required result.

Theorem 3.10 *The compression spectrum of the operator* $B(r,s,t)$ *over is given by*

$$\sigma_{co}\left(B(r,s,t),cs\right) = S_1.$$

Proof: By proposition 2.1 (e), we get $\sigma_p\left(B(r,s,t)^*,cs^*\right) = \sigma_{co}\left(B(r,s,t),cs\right)$.
Using Theorem 3.4, we get the required result.

Theorem 3.11 *The defect spectrum of the operator* $B(r,s,t)$ *over is given by*

$$\sigma_\delta\left(B(r,s,t),cs\right) = S.$$

Proof: From Table 2, we have $\sigma_{\delta}\left(B(r,s,t),cs\right) = \sigma\left(B(r,s,t),cs\right)\setminus I_3\sigma\left(B(r,s,t),cs\right)$. Also, $\sigma_p\left(B(r,s,t),cs\right) = I_3\sigma\left(B(r,s,t),cs\right)\cup II_3\sigma\left(B(r,s,t),cs\right)\cup III_3\sigma\left(B(r,s,t),cs\right)$. By Theorem 3.3, we have $\sigma_p\left(B(r,s,t),cs\right)=\varnothing$ and so $I_3\sigma\left(B(r,s,t),cs\right)=\varnothing$.

Hence $\sigma_{\delta}\left(B(r,s,t),cs\right)=S$.

COROLLARY 3.13 *The following statements hold:*

(i) $\sigma_{ap}\left(B(r,s,t)^{*},cs^{*}\cong bv\right)=S$.

(ii) $\sigma_{\delta}\left(B(r,s,t)^{*},cs^{*}\cong bv\right)=\begin{cases} S\setminus\{r\}, & if\ |t|<|s| \\ S, & if\ |t|\geq|s| \end{cases}$.

Proof: Using Proposition 2.1 (c) and (d), we get

$$\sigma_{ap}\left(B(r,s,t)^{*},cs^{*}\cong bv\right)=\sigma_{\delta}\left(B(r,s,t),cs\right) \text{ and}$$

$$\sigma_{\delta}\left(B(r,s,t)^{*},cs^{*}\cong bv\right)=\sigma_{ap}\left(B(r,s,t),cs\right).$$

Using Theorem 3.9 and Theorem 3.11, we get the required results.

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