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Confidence intervals for functions of coefficients of variation with bounded parameter spaces in two gamma distributions

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Abstract

The problem of estimating parameters in a gamma distribution has been widely studied with respect to both theories and applications. In special cases, when the parameter space is bounded, the construction of the confidence interval based on the classical Neyman procedure is unsatisfactory because the information regarding the restriction of the parameter is disregarded. In order to develop the estimator for this issue, the confidence intervals for the coefficient of variation for the case of a gamma distribution were proposed. Extending to two populations, the confidence intervals for the difference and the ratio of coefficients of variation with restricted parameters were presented. Monte Carlo simulations were used to investigate the performance of the proposed estimators. The results showed that the proposed confidence intervals performed better than the compared estimators in terms of expected length, especially when the coefficients of variation were close to the boundary. Additionally, two examples using real data were analyzed to illustrate the findings of the paper.

Keywords: bounded parameter space, coefficient of variation, confidence interval, gamma distribution, simulation

1. Introduction

Interval estimation is a common approach for estimating the parameter of interest. Since it is guaranteed by the confidence level that the unknown parameter is contained in the confidence interval with common probability, the interval estimator is more meaningful, and it provides more information with respect to the parameter than the point estimator (Casella & Berger, 2002). In frequentist theory, when lacking a priori knowledge of the parameter, the confidence interval is usually derived from the classical Neyman procedure. That means statistical inference based on that traditional approach is available for the natural parameter space (Mandelkern, 2002). However, in fact, the bounded parameter is found in many practical applications, such as engineering process controls, health science, and physical experiments. In this case, the crucial problem of the confidence interval obtained by the Neyman procedure is that when the confidence interval partly or completely departs from the permissible range for the parameter, it invalidates the assertion of the $(1-\gamma)100\%$ confidence interval (Fraser *et al.*, 2004). Therefore, alternative approaches have been discussed for obtaining accurate confidence intervals. Recent work related to statistical inference for bounded parameters is as follows.

In the paper of Wang (2008), the confidence intervals for the normal mean in cases where the parameter space is bounded were derived using the rp interval, Bayesian interval, and likelihood interval. These confidence intervals were compared with the standard confidence interval and the minimax interval by simulation. It was found that, although the coverage probabilities of the standard confidence interval were lower than those of the rp interval and Bayesian interval, they were greater than the nominal coverage level and simpler to use in practice. The standard confidence interval also provided the short length interval. Furthermore, the standard confidence intervals with bounded parameters in the normal

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distribution were studied by Niwitpong (2011) and Sappakitkamjorn and Niwitpong (2013). The results via Monte Carlo simulation showed that their confidence intervals performed well in terms of coverage probability and expected length.

In a skewed distribution, Eeden (1995) introduced minimax estimation for scale invariant square error loss when the scale parameter is bounded below. Chang (2010) presented the admissible estimators of the restricted scale parameters in the gamma distributions. Using the rp interval, Wang (2012) studied the confidence intervals for the means with bounded parameter spaces in the exponential families. Moreover, Niwitpong (2013a, 2013b) proposed standard confidence intervals for the mean and the coefficient of variation in a lognormal distribution with restricted parameter space, and then developed to the standard confidence intervals for the difference and the ratio of two lognormal means (Niwitpong, 2015). The results showed that the confidence intervals of Niwitpong (2015) performed well in terms of coverage probability and expected length.

As the reviewed literature indicates, the restriction parameter has been studied with both normal and skewed distributions. Thus, it is also likely that the parameter space may be bounded in the gamma distribution. Its probability density function is given in (1). This distribution is applied in actuarial science and many fields of applied statistics as the waiting time until α th event occurs. In this study, we focus on the confidence intervals for functions of coefficients of variation in two gamma distributions when the parameter spaces are restricted. The coefficient of variation is a statistical measurement used to report the dispersion of variables, and it can be applied to compare several variables expressed in different units. In the gamma distribution, the coefficient of variation is the function of only one parameter, the shape parameter, while its variance depends on both the shape and scale parameters. This is the reason for considering the statistical inference of the coefficient of variation. In this work, the coverage probabilities and expected lengths of the proposed and the existing confidence intervals are studied through Monte Carlo simulations. Moreover, we use real-world examples to illustrate the confidence interval proposed in this paper.

2. Confidence Intervals for the Coefficient of Variation of the Gamma Distribution

In this section, we explain the methods for constructing the confidence intervals for the single coefficient of variation. The criterion of the study is as follows. Let $X = (X_1, X_2, \dots, X_n)$ be a random sample from the gamma distribution with the shape parameter α_1 and scale parameter β_1, denoted as $X \sim \text{Gamma}(\alpha_1, \beta_1)$. The probability density function of X is given by

$$f_X(x;\alpha_1,\beta_1) = \begin{cases} \dfrac{1}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} x^{\alpha_1 - 1}\exp\{-x/\beta_1\} & ; \quad 0 \le x < \infty \\ 0 & ; \quad x < 0 \end{cases}, \quad (1)$$

where $\alpha_1 \in \Theta_1$ and $\beta_1 \in \Omega_1$. $\Theta_1 = \{\alpha_1 : \alpha_1 > 0\}$ and $\Omega_1 = \{\beta_1 : \beta_1 > 0\}$ are the natural parameter spaces. The mean and variance of X are $E(X) = \alpha_1\beta_1$ and $Var(X) = \alpha_1\beta_1^2$, respectively. Thus, the coefficient of variation of X is given by $\tau_1 = 1/\sqrt{\alpha_1}$. Since α_1 is the unknown parameter, it is required to be estimated.

We first consider the maximum likelihood estimators for α_1 and β_1. From the density shown in (1), the log-likelihood function of α_1 and β_1 is given by

$$\ln L(\alpha_1,\beta_1) = -\sum_{i=1}^{n}\frac{X_i}{\beta_1} + (\alpha_1 - 1)\sum_{i=1}^{n}\ln X_i - n\ln\Gamma(\alpha_1) - n\alpha_1\ln\beta_1 .$$

Taking partial derivatives of the above equation with respect to α_1 and β_1, respectively, the score function is derived as

$$U(\alpha_1,\beta_1) = \begin{bmatrix} \sum_{i=1}^{n}\ln X_i - n\ln\alpha_1 + n/(2\alpha_1) - n\ln\beta_1 \\ \sum_{i=1}^{n} X_i/\beta_1^2 - n\alpha_1/\beta_1 \end{bmatrix}.$$

Then, we yield the maximum likelihood estimators for α_1 and β_1, respectively,

$$\hat{\alpha}_1 = \frac{1}{2(\ln\bar{X} - \sum_{i=1}^{n}\ln X_i / n)}$$

$$\hat{\beta}_1 = \frac{\bar{X}}{\hat{\alpha}_1},$$

where $\bar{X} = \sum_{i=1}^{n} X_i / n$ is the sample mean of X. Also, the sample coefficient of variation for τ_1 is given by $\hat{\tau}_1 = 1/\sqrt{\hat{\alpha}_1}$.

Next, the confidence intervals for τ_1 using two methods, the score and the Wald intervals are investigated. These approaches are considered later.

2.1 Confidence interval based on the score method

Let α_1 and β_1 be the parameter of interest and the nuisance parameter, respectively. In general, the score or Rao statistic is denoted as

$$W_u = U^T(\alpha_0,\hat{\beta}_0) I^{-1}(\alpha_0,\hat{\beta}_0)\, U(\alpha_0,\hat{\beta}_0),$$

where $\hat{\beta}_0$ is the maximum likelihood estimator for β_1 under the null hypothesis $H_0 : \alpha_1 = \alpha_0$, $U(\alpha_0,\hat{\beta}_0)$ is the vector of the score function, and $I(\alpha_0,\hat{\beta}_0)$ is the matrix of the Fisher information. Here, it is easy to derive that the score function under H_0 is

$$U(\alpha_0,\hat{\beta}_0) = \begin{bmatrix} \sum_{i=1}^{n}\ln X_i + n/(2\alpha_0) - n\ln\bar{X} \\ 0 \end{bmatrix}.$$

The inverse of the Fisher information can be derived as

$$I^{-1}(\alpha_0,\hat{\beta}_0) = \begin{bmatrix} 2\alpha_0^2/n & -2\bar{X}/n \\ -2\bar{X}/n & \bar{X}^2(2\alpha_0+1)/(n\alpha_0^3) \end{bmatrix}.$$

Using the property of the score function, we can see that the pivotal

$$Z_{score} = \sqrt{\frac{2\alpha_0^2}{n}}\left(\sum_{i=1}^{n}\ln X_i + \frac{n}{2\alpha_0} - n\ln\overline{X}\right) \quad (2)$$

converges in distribution to the standard normal distribution. Since the variance of $\hat{\alpha}_1$ is $2\alpha_0^2 / n$, we approximate it by substituting $\hat{\alpha}_1$ in its variance. Under H_0, statistic in (2) is given as

$$Z_{score} \cong \sqrt{\frac{2\hat{\alpha}_1^2}{n}}\left(\sum_{i=1}^{n}\ln X_i + \frac{n}{2\alpha_1} - n\ln\overline{X}\right).$$

From the probability statement, $1-\gamma = P(-Z_{\gamma/2} \le Z_{score} \le Z_{\gamma/2})$, it can be simply written as $1-\gamma = P(l_{s1} \le \tau_1 \le u_{s1})$. Therefore, the $(1-\gamma)100\%$ confidence interval for τ_1 based on the score method, CI_s, is given by

$$[l_{s1}, u_{s1}] = \left[\sqrt{\frac{2}{n}\left(z_1 - Z_{\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}_1^2}}\right)}, \sqrt{\frac{2}{n}\left(z_1 + Z_{\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}_1^2}}\right)}\right], \quad (3)$$

where $z_1 = n\ln\overline{X} - \sum_{i=1}^{n}\ln X_i$ and $Z_{\gamma/2}$ is the $(\gamma/2)100$ th percentile of the standard normal distribution.

2.2 Confidence interval based on the Wald method

The Wald statistic is an asymptotic statistic derived from the property of the maximum likelihood estimator. The general form of the Wald statistic under the null hypothesis $H_0: \alpha_1 = \alpha_0$ is defined as

$$W_e = (\hat{\alpha}_1 - \alpha_0)^T [I^{\alpha_1\alpha_1}(\hat{\alpha}_1, \hat{\beta}_1)]^{-1}(\hat{\alpha}_1 - \alpha_0),$$

where $I^{\alpha_1\alpha_1}(\hat{\alpha}_1, \hat{\beta}_1)$ is the estimated variance of $\hat{\alpha}_1$ obtained from the first row and the first column of $I^{-1}(\hat{\alpha}_1, \hat{\beta})$. Using the information of partial derivatives from the previous sub-section, the inverse matrix is given by

$$I^{-1}(\hat{\alpha}_1, \hat{\beta}_1) = \begin{bmatrix} 2\hat{\alpha}_1^2 / n & -2\overline{X}/n \\ -2\overline{X}/n & \overline{X}^2(2\hat{\alpha}_1 + 1)/(n\hat{\alpha}_1^3) \end{bmatrix}$$

with $I^{\alpha_1\alpha_1}(\hat{\alpha}_1, \hat{\beta}_1) = 2\hat{\alpha}_1^2 / n$. Therefore, under H_0, we obtain the Wald statistic

$$Z_{wald} \cong \sqrt{\frac{n}{2\hat{\alpha}_1^2}}(\hat{\alpha}_1 - \alpha_1),$$

which has the limiting distribution of standard normal distribution. Therefore, the $(1-\gamma)100\%$ confidence interval for τ_1 based on the Wald method, CI_w, is given by

$$[l_{w1}, u_{w1}] = \left[1/\sqrt{\hat{\alpha}_1 + Z_{\gamma/2}\sqrt{\frac{2\hat{\alpha}_1^2}{n}}}, 1/\sqrt{\hat{\alpha}_1 - Z_{\gamma/2}\sqrt{\frac{2\hat{\alpha}_1^2}{n}}}\right]. \quad (4)$$

In addition, suppose that $Y = (Y_1, Y_2, \ldots, Y_m)$ be a random sample, where $Y \sim \text{Gamma}(\alpha_2, \beta_2)$. The coefficient of variation of Y is $\tau_2 = 1/\sqrt{\alpha_2}$ with the point estimator $\hat{\tau}_2 = 1/\sqrt{\hat{\alpha}_2}$. Also, we have the confidence intervals for τ_2 based on the score method, $[l_{s2}, u_{s2}]$, and the Wald method, $[l_{w2}, u_{w2}]$. Note that these confidence intervals are similar to (3) and (4), except that they use the information from Y.

Extending the problem of this section, the confidence intervals for the single coefficient of variation in the gamma distribution with bounded parameter space are investigated in the next section.

3. Confidence Intervals for a Bounded Coefficient of Variation in the Gamma Distribution

The method for constructing the confidence intervals for bounded parameter space applied in this paper is the standard approach. It is derived using the intersection of the general confidence limits and the bounds of parameter space. For this method, the information concerning the restriction of the parameter is used, in contrast to the classical Neyman approach. Following Wang (2008), when a parameter ε is bounded between values c and d, the standard confidence interval for ε is

$$CI = [\max(c, l_\varepsilon), \min(d, u_\varepsilon)], \quad (5)$$

where l_ε and u_ε are the lower and upper limits of the general confidence interval for ε, respectively. Obviously, we have four cases as follows:

(i) If $c > l_\varepsilon$ and $d < u_\varepsilon$, then $CI = [c, d]$

(ii) If $c > l_\varepsilon$ and $d > u_\varepsilon$, then $CI = [c, u_\varepsilon]$

(iii) If $c < l_\varepsilon$ and $d < u_\varepsilon$, then $CI = [l_\varepsilon, d]$

(iv) If $c < l_\varepsilon$ and $d > u_\varepsilon$, then $CI = [l_\varepsilon, u_\varepsilon]$.

Note that the confidence interval obtained from the above cases has the shortest length interval.

The procedure for constructing the confidence intervals for τ_1 with parameter restrictions is described later. Assume that the parameter space of α_1 is known to be restricted and bounded between values $a_1^{'}$ and $b_1^{'}$, where $0 < a_1^{'} < b_1^{'}$. Since τ_1 is a function of α_1, when α_1 is bounded, τ_1 is also bounded. It can be simply written as

$$a_1^{'} < \alpha_1 < b_1^{'}$$

$$\sqrt{a_1^{'}} < \sqrt{\alpha_1} < \sqrt{b_1^{'}}$$

$$a_{R1} = 1/\sqrt{b_1^{'}} < \tau_1 < 1/\sqrt{a_1^{'}} = b_{R1}.$$

Using the confidence intervals for τ_1 presented in the previous section and the information of the restriction, the $(1-\gamma)100\%$ confidence intervals using the score and the Wald intervals for τ_1 with bounded parameter space are given as

$$CI_{sR} = [\max(a_{R1}, l_{s1}), \min(b_{R1}, u_{s1})] \quad (6)$$

$$CI_{wR} = [\max(a_{R1}, l_{w1}), \min(b_{R1}, u_{w1})], \quad (7)$$

respectively. We also note that l_{s1} and u_{s1}, and l_{w1} and u_{w1} are the general confidence limits obtained from (3) and (4), respectively.

4. Confidence Intervals for the Difference of Coefficients of Variation in the Gamma Distributions

In this section, we follow the confidence intervals for the difference of coefficients of variation presented in the paper of Sangnawakij and Niwitpong (2015). The notations are given at the start. Suppose that X and Y be two random samples with $X \sim \text{Gamma}(\alpha_1, \beta_1)$ and $Y \sim \text{Gamma}(\alpha_2, \beta_2)$. Also, X and Y are independent. The difference of coefficients of variation is defined as $\psi = \tau_1 - \tau_2$.

They introduced the confidence intervals for ψ using the method of variance of estimates recovery (MOVER) with the score interval, CI_{ds}, and Wald interval, CI_{dw}. These confidence intervals are given by

$$[l_{ds}, u_{ds}] =$$

$$\left[\hat{\psi} - \sqrt{(\hat{\tau}_1 - l_{s1})^2 + (u_{s2} - \hat{\tau}_2)^2} \ , \ \hat{\psi} + \sqrt{(u_{s1} - \hat{\tau}_1)^2 + (\hat{\tau}_2 - l_{s2})^2} \ \right] \quad (8)$$

$$[l_{dw}, u_{dw}] =$$

$$\left[\hat{\psi} - \sqrt{(\hat{\tau}_1 - l_{w1})^2 + (u_{w2} - \hat{\tau}_2)^2} \ , \ \hat{\psi} + \sqrt{(u_{w1} - \hat{\tau}_1)^2 + (\hat{\tau}_2 - l_{w2})^2} \ \right] \quad (9)$$

where $\hat{\psi} = \hat{\tau}_1 - \hat{\tau}_2$ is the sample difference of coefficients of variation.

In the simulation, it was found that CI_{ds} and CI_{dw} of Sangnawakij and Niwitpong (2015) performed well in terms of coverage probability in almost all cases, and the lengths of CI_{ds} were slightly shorter than those of CI_{dw}. Thus, in the next section, we use these estimators to develop the confidence interval for the difference of coefficients of variation with bounded parameter spaces.

5. Confidence Intervals for the Difference of Coefficients of Variation with Bounded Parameters in the Gamma Distributions

Here, we are interested in the restriction of parameters in the gamma distributions in order to construct the confidence intervals for the difference of coefficients of variation. Assume that the shape parameters α_1 and α_2 are bounded, $a_1^{'} < \alpha_1 < b_1^{'}$ and $a_2^{'} < \alpha_2 < b_2^{'}$, where $0 < a_i^{'} < b_i^{'}$ for i = 1, 2. Thus, we have

$$\sqrt{a_i^{'}} < \sqrt{\alpha_i} < \sqrt{b_i^{'}}$$

$$a_{Ri} = 1/\sqrt{b_i^{'}} < \tau_i < 1/\sqrt{a_i^{'}} = b_{Ri}.$$

Since the difference of coefficients of variation ψ is a function of parameters α_1 and α_2 which are bounded, it is also bounded as

$$1/\sqrt{b_1^{'}} - 1/\sqrt{a_2^{'}} < \tau_1 - \tau_2 < 1/\sqrt{a_1^{'}} - 1/\sqrt{b_2^{'}}$$

$$a_{DR} = a_{R1} - b_{R2} < \psi < b_{R1} - a_{R2} = b_{DR},$$

where $a_{R1} = 1/\sqrt{b_1^{'}}$, $b_{R1} = 1/\sqrt{a_1^{'}}$, $a_{R2} = 1/\sqrt{b_2^{'}}$, and $b_{R2} = 1/\sqrt{a_2^{'}}$.

Using the standard approach, the $(1-\gamma)100\%$ confidence intervals for ψ with bounded parameters based on the score interval and the Wald interval are

$$CI_{sDR} = [\max(a_{DR}, l_{ds}), \min(b_{DR}, u_{ds})] \quad (10)$$

$$CI_{wDR} = [\max(a_{DR}, l_{dw}), \min(b_{DR}, u_{dw})], \quad (11)$$

respectively. Note that l_{ds} and u_{ds}, and l_{dw} and u_{dw} are the general confidence limits for ψ obtained from (8) and (9), respectively.

6. Confidence Intervals for the Ratio of Coefficients of Variation in the Gamma Distributions

Let X and Y be two random samples from the gamma distributions as mentioned in Section 4. Here, the ratio of coefficients of variation is $\eta = \tau_1 / \tau_2$. Recently, Sangnawakij *et al.* (2015) introduced the confidence intervals for η based on the MOVER with the score interval, CI_{rs}, and the Wald interval, CI_{rw}, where

$$[l_{rs}, u_{rs}] =$$

$$\left[\left(\hat{\tau}_1\hat{\tau}_2 - \sqrt{(\hat{\tau}_1\hat{\tau}_2)^2 - l_{s1}u_{s2}(2\hat{\tau}_2 - u_{s2})(2\hat{\tau}_1 - l_{s1})} \right) / \left(u_{s2}(2\hat{\tau}_2 - u_{s2}) \right), \right.$$

$$\left. \left(\hat{\tau}_1\hat{\tau}_2 + \sqrt{(\hat{\tau}_1\hat{\tau}_2)^2 - l_{s2}u_{s1}(2\hat{\tau}_2 - l_{s2})(2\hat{\tau}_1 - u_{s1})} \right) / \left(l_{s2}(2\hat{\tau}_2 - l_{s2}) \right) \right] \quad (12)$$

$$[l_{rw}, u_{rw}] =$$

$$\left[\left(\hat{\tau}_1\hat{\tau}_2 - \sqrt{(\hat{\tau}_1\hat{\tau}_2)^2 - l_{w1}u_{w2}(2\hat{\tau}_2 - u_{w2})(2\hat{\tau}_1 - l_{w1})} \right) / \left(u_{w2}(2\hat{\tau}_2 - u_{w2}) \right), \right.$$

$$\left. \left(\hat{\tau}_1\hat{\tau}_2 + \sqrt{(\hat{\tau}_1\hat{\tau}_2)^2 - l_{w2}u_{w1}(2\hat{\tau}_2 - l_{w2})(2\hat{\tau}_1 - u_{w1})} \right) / \left(l_{w2}(2\hat{\tau}_2 - l_{w2}) \right) \right], \quad (13)$$

respectively. In the simulation of Sangnawakij *et al.* (2015), it was found that the coverage probabilities of CI_{rs} and CI_{rw} satisfied the nominal coverage level and performed well in terms of expected length in all cases. Therefore, these two existing estimators are considered to construct the new confidence intervals in the next section.

7. Confidence Intervals for the Ratio of Coefficients of Variation with Bounded Parameters in the Gamma Distributions

Suppose that parameters α_1 and α_2 are bounded as $a_1^{'} < \alpha_1 < b_1^{'}$ and $a_2^{'} < \alpha_2 < b_2^{'}$. Using the information regarding the restriction of α_1 and α_2 in Section 5, the ratio of coefficients of variation in two gamma distributions is bounded as

$$\sqrt{a_2^{'}} / \sqrt{b_1^{'}} < \tau_1 / \tau_2 < \sqrt{b_2^{'}} / \sqrt{a_1^{'}}$$

$$a_{RR} = a_{R1} / b_{R2} < \eta < b_{R1} / a_{R2} = b_{RR}.$$

Hence, it is easy to see that the $(1-\gamma)100\%$ standard confidence interval for η with bounded parameters based on the score interval and the Wald interval are

$$CI_{sRR} = [\max(a_{RR}, l_{rs}), \min(b_{RR}, u_{rs})] \quad (14)$$

$$CI_{wRR} = [\max(a_{RR}, l_{rw}), \min(b_{RR}, u_{rw})], \quad (15)$$

respectively, where l_{rs} and u_{rs}, and l_{rw} and u_{rw} are the general confidence limits for η obtained from (12) and (13), respectively.

8. Simulation Studies

In this study, the performance of the proposed confidence intervals is investigated using Monte Carlo simulation. The simulations are done using the R statistical program (Venables *et al.*, 2015) with M = 10,000 replications in each case. The estimated coverage probability (CP) and the estimated expected length (EL), respectively, are given by

$$\text{CP} = \frac{c(L \le \varepsilon \le U)}{M} \quad \text{and} \quad \text{EL} = \frac{\sum_{h=1}^{M}(U_h - L_h)}{M},$$

where $c(L \le \varepsilon \le U)$ is the number of simulation runs when parameter ε lies within the confidence interval. Here, we choose a confidence interval which has a coverage probability greater than or close to the nominal coverage level, and short length interval.

For one population, the data are generated from a gamma distribution with $\beta = 2$ and α is adjusted to get the required coefficient of variation τ. For the restriction at $0.05 < \tau < 0.51$, we set τ = 0.05, 0.10, 0.20, 0.28, 0.30, 0.33, 0.35, 0.40, 0.45, 0.47, 0.49, and 0.50. The sample sizes are chosen to be n = 10, 30, 50, 100, and 200. Then, the performance of 95% confidence intervals for τ is computed.

For two populations, the data are generated from two independent gamma distributions with (α_i, β_i) where β_i are fixed at 2, and α_i are adjusted to yield the required coefficients of variation, which is computed by $\alpha_i = 1/\tau_i^2$, for n = 1, 2. For the restriction at $0.05 < \tau_1 < 0.51$ and $0.05 < \tau_2 < 0.51$, the coefficients of variation are set at (τ_1, τ_2) = (0.05,0.05), (0.10,0.49), (0.15,0.45), (0.28,0.28), (0.45,0.15), (0.49,0.10), and (0.50,0.05). Next, the coverage probabilities and expected lengths of the 95% confidence intervals for the difference of coefficients of variation ψ and the ratio of coefficients of variation η are evaluated. The performance of all proposed confidence intervals is also compared with that of the existing confidence intervals. The simulation results are described in the next section.

9. Results and Discussion

We first consider the performance of the confidence intervals for τ with bounded parameter space. The results are shown in Table 1. For $10 \le n \le 50$, CI_{sR} provides coverage probabilities less than the nominal coverage level at 0.95. However, when the sample size increases, the coverage probabilities of CI_{sR} increase, and are greater than 0.95.

Table 1. The coverage probabilities and expected lengths of the 95% confidence intervals for the coefficient of variation, when $0.05 < \tau < 0.51$

n	τ	Coverage probability		Expected length			
		CI_s, CI_{sR}	CI_w, CI_{wR}	CI_s	CI_w	CI_{sR}	CI_{wR}
10	0.05	0.8141	0.9717	0.0471	0.0979	0.0134	0.0816
	0.10	0.8033	0.9695	0.0942	0.1957	0.0767	0.1947
	0.20	0.8024	0.9696	0.1883	0.3912	0.1871	0.3327
	0.28	0.8153	0.9684	0.2658	0.5522	0.2640	0.3135
	0.30	0.8105	0.9680	0.2840	0.5900	0.2798	0.3028
	0.33	0.8091	0.9684	0.3123	0.6488	0.3012	0.2842
	0.35	0.8131	0.9704	0.3310	0.6876	0.3126	0.2715
	0.40	0.8120	0.9643	0.3799	0.7893	0.3301	0.2372
	0.45	0.8258	0.9638	0.4322	0.8979	0.3349	0.2000
	0.47	0.8269	0.9626	0.4516	0.9382	0.3334	0.1862
	0.49	0.8234	0.9672	0.4702	0.9768	0.3318	0.1729
	0.50	0.8265	0.9606	0.4802	0.9976	0.3299	0.1657

Table 1. Continued

n	τ	Coverage probability		Expected length			
		CI_s , CI_{sR}	CI_w , CI_{wR}	CI_s	CI_w	CI_{sR}	CI_{wR}
30	0.05	0.8955	0.9604	0.0255	0.0296	0.0098	0.0192
	0.10	0.8948	0.9566	0.0512	0.0594	0.0511	0.0593
	0.20	0.8983	0.9553	0.1026	0.1190	0.1026	0.1190
	0.28	0.9024	0.9558	0.1441	0.1670	0.1441	0.1668
	0.30	0.9042	0.9556	0.1547	0.1793	0.1546	0.1779
	0.33	0.9093	0.9540	0.1704	0.1976	0.1700	0.1897
	0.35	0.9042	0.9559	0.1806	0.2094	0.1791	0.1923
	0.40	0.9091	0.9556	0.2071	0.2401	0.1919	0.1781
	0.45	0.9140	0.9551	0.2340	0.2715	0.1828	0.1441
	0.47	0.9158	0.9534	0.2451	0.2834	0.1735	0.1290
	0.49	0.9163	0.9511	0.2555	0.2961	0.1628	0.1126
	0.50	0.9198	0.9498	0.2611	0.3024	0.1566	0.1044
50	0.05	0.9153	0.9581	0.0197	0.0214	0.0080	0.0130
	0.10	0.9196	0.9534	0.0395	0.0429	0.0395	0.0429
	0.20	0.9190	0.9521	0.0792	0.0861	0.0792	0.0861
	0.28	0.9253	0.9539	0.1110	0.1206	0.1110	0.1206
	0.30	0.9276	0.9524	0.1193	0.1296	0.1193	0.1296
	0.33	0.9209	0.9508	0.1312	0.1428	0.1311	0.1424
	0.35	0.9289	0.9535	0.1394	0.1516	0.1392	0.1498
	0.45	0.9302	0.9494	0.1803	0.1958	0.1464	0.1251
	0.47	0.9371	0.9540	0.1883	0.2050	0.1366	0.1092
	0.49	0.9386	0.9560	0.1972	0.2139	0.1227	0.0927
	0.50	0.9400	0.9474	0.2011	0.2190	0.1162	0.0828
100	0.05	0.9335	0.9496	0.0139	0.0145	0.0061	0.0084
	0.10	0.9345	0.9512	0.0278	0.0289	0.0278	0.0289
	0.20	0.9352	0.9510	0.0557	0.0580	0.0557	0.0580
	0.28	0.9444	0.9501	0.0783	0.0815	0.0783	0.0815
	0.30	0.9463	0.9514	0.0840	0.0874	0.0840	0.0874
	0.33	0.9455	0.9504	0.0925	0.0963	0.0925	0.0963
	0.35	0.9436	0.9466	0.0982	0.1022	0.0982	0.1022
	0.40	0.9488	0.9462	0.1127	0.1172	0.1121	0.1149
	0.45	0.9475	0.9468	0.1270	0.1322	0.1112	0.1026
	0.47	0.9507	0.9441	0.1330	0.1382	0.1007	0.0882
	0.49	0.9533	0.9470	0.1390	0.1447	0.0861	0.0700
	0.50	0.9525	0.9413	0.1419	0.1475	0.0780	0.0618
200	0.05	0.9414	0.9539	0.0098	0.0100	0.0044	0.0055
	0.10	0.9418	0.9523	0.0196	0.0200	0.0196	0.0200
	0.20	0.9463	0.9507	0.0394	0.0402	0.0394	0.0402
	0.28	0.9502	0.9509	0.0553	0.0564	0.0553	0.0564
	0.30	0.9480	0.9508	0.0593	0.0605	0.0593	0.0605
	0.33	0.9508	0.9487	0.0653	0.0666	0.0653	0.0666
	0.35	0.9519	0.9467	0.0695	0.0708	0.0695	0.0708
	0.40	0.9501	0.9428	0.0796	0.0810	0.0796	0.0810
	0.45	0.9543	0.9400	0.0898	0.0914	0.0846	0.0825
	0.47	0.9514	0.9430	0.0939	0.0957	0.0768	0.0710
	0.49	0.9525	0.9410	0.0981	0.0998	0.0622	0.0553
	0.50	0.9531	0.9460	0.1002	0.1018	0.0535	0.0461

Meanwhile, the coverage probabilities of CI_{wR} satisfy the nominal coverage level. Since the coverage probabilities of CI_s and CI_{sR}, CI_w and CI_{wR}, respectively, provide the same results, the performance of those confidence intervals is appraised in terms of the expected length. It was found that the expected lengths of CI_{sR} and CI_{wR} are shorter than those of CI_s and CI_w in almost all cases. In addition, the expected lengths of CI_{sR} are slightly smaller than those of CI_{wR} for large τ. From Table 2, it can be seen that the ratios of expected length between CI_s and CI_{sR}, and CI_w and CI_{wR}, respectively, are greater than one, when τ is close to the boundary of (0.05,0.50). That means the confidence intervals obtained from the method involving a bounded interval more accurately cover the true parameter than confidence intervals

Table 2. The ratio of expected length of the 95% confidence intervals for the coefficient of variation, when $0.05 < \tau < 0.51$

τ	$\frac{E(CI_s)}{E(CI_{sR})}$	$\frac{E(CI_w)}{E(CI_{wR})}$	$\frac{E(CI_{sR})}{E(CI_{wR})}$	τ	$\frac{E(CI_s)}{E(CI_{sR})}$	$\frac{E(CI_w)}{E(CI_{wR})}$	$\frac{E(CI_{sR})}{E(CI_{wR})}$
	$n = 10$				$n = 100$		
0.05	3.5192	1.1999	0.1641	0.05	2.2716	1.7271	0.7305
0.10	1.2289	1.0054	0.3938	0.10	1.0000	1.0000	0.9608
0.20	1.0063	1.1761	0.5626	0.20	1.0000	1.0000	0.9608
0.28	1.0068	1.7614	0.8422	0.28	1.0000	1.0000	0.9608
0.30	1.0149	1.9486	0.9242	0.30	1.0000	1.0000	0.9608
0.33	1.0369	2.2831	1.0598	0.33	1.0000	1.0000	0.9608
0.35	1.0590	2.5331	1.1514	0.35	1.0000	1.0000	0.9609
0.40	1.1510	3.3275	1.3917	0.40	1.0057	1.0198	0.9754
0.45	1.2906	4.4897	1.6746	0.45	1.1427	1.2884	1.0833
0.47	1.3546	5.0384	1.7904	0.47	1.3208	1.5669	1.1420
0.49	1.4170	5.6486	1.9188	0.49	1.6144	2.0671	1.2300
0.50	1.4558	6.0191	1.9902	0.50	1.8197	2.3867	1.2620
	$n = 30$				$n = 200$		
0.05	2.6180	1.5398	0.5073	0.05	2.2089	1.8181	0.8071
0.10	1.0008	1.0001	0.8618	0.10	1.0000	1.0000	0.9806
0.20	1.0000	1.0000	0.8625	0.20	1.0000	1.0000	0.9806
0.28	1.0000	1.0016	0.8639	0.28	1.0000	1.0000	0.9810
0.30	1.0003	1.0079	0.8691	0.30	1.0000	1.0000	0.9800
0.33	1.0023	1.0415	0.8962	0.33	1.0000	1.0000	0.9806
0.35	1.0085	1.0888	0.9312	0.35	1.0000	1.0000	0.9806
0.40	1.0789	1.3482	1.0777	0.40	1.0001	1.0000	0.9826
0.45	1.2797	1.8841	1.2688	0.45	1.0619	1.1079	1.0249
0.47	1.4126	2.1969	1.3449	0.47	1.2220	1.3486	1.0829
0.49	1.5697	2.6297	1.4458	0.49	1.5777	1.8047	1.1242
0.50	1.6677	2.8966	1.4995	0.50	1.8719	2.2082	1.1607
	$n = 50$						
0.05	2.4494	1.6403	0.6161				
0.10	1.0000	1.0000	0.9200				
0.20	1.0000	1.0000	0.9200				
0.28	1.0000	1.0000	0.9200				
0.30	1.0000	1.0000	0.9208				
0.33	1.0002	1.0028	0.9209				
0.35	1.0014	1.0120	0.9290				
0.45	1.2314	1.5651	1.1701				
0.47	1.3785	1.8773	1.2508				
0.49	1.6064	2.3074	1.3241				
0.50	1.7312	2.6449	1.4028				

derived by the classical procedure.

The results of the confidence intervals for ψ with bounded parameter spaces are presented in Table 3. For $10 \le n,m \le 30$, CI_{sDR} yields coverage probabilities less than 0.95. However, when the sample size increases, it has coverage probabilities greater than or close to 0.95. The coverage probabilities of CI_{wDR} satisfy the nominal coverage in general, expect for sample sizes equal to 200. The results are similar to those of Sangnawakij and Niwitpong (2015). Furthermore, the expected lengths of CI_{sDR} and CI_{wDR} are shorter than those of CI_{ds} and CI_{dw}. As can be seen from Table 4, the ratios of expected length of CI_{ds} and CI_{sDR} , and CI_{dw} and CI_{wDR} , respectively, are greater than one, especially, when τ_1 and τ_2 are close to the boundary of (0.50,0.05). Also, the expected lengths of CI_{sDR} are also longer than those of CI_{wDR} .

Finally, the performance of the confidence intervals for η with bounded parameter spaces is considered. The results from Table 5 show that CI_{sRR} and CI_{wRR} provide coverage probabilities greater than or close to 0.95. In general, the coverage probabilities of CI_{wRR} are higher than those of CI_{sRR} . The results are also similar to those of Sangnawakij

Table 3. The coverage probabilities and expected lengths of the 95% confidence intervals for the difference of coefficients of variation, when $0.05 < \tau_1, \tau_2 < 0.51$

(n,m)	(τ_1,τ_2)	Coverage probability		Expected length			
		CI_{ds} , CI_{sDR}	CI_{dw} , CI_{wDR}	CI_{ds}	CI_{dw}	CI_{sDR}	CI_{wDR}
(10,10)	(0.05,0.50)	0.8358	0.9732	0.4841	1.0267	0.3280	0.1895
	(0.10,0.49)	0.8603	0.9858	0.4848	1.0691	0.3692	0.3044
	(0.15,0.45)	0.8875	0.9933	0.4601	1.0602	0.4034	0.4591
	(0.28,0.28)	0.9963	0.9973	0.3911	0.9679	0.3910	0.7958
	(0.45,0.15)	0.8986	0.9897	0.4623	1.0646	0.4047	0.4561
	(0.49,0.10)	0.8647	0.9866	0.4832	1.0655	0.3697	0.3047
	(0.50,0.05)	0.8398	0.9703	0.4852	1.0290	0.3276	0.1888
(30,30)	(0.05,0.50)	0.9195	0.9454	0.2623	0.3050	0.1560	0.1053
	(0.10,0.49)	0.9266	0.9514	0.2612	0.3065	0.2030	0.1674
	(0.15,0.45)	0.9336	0.9570	0.2471	0.2940	0.2360	0.2433
	(0.28,0.28)	0.9683	0.9700	0.2062	0.2541	0.2062	0.2541
	(0.45,0.15)	0.9361	0.9611	0.2469	0.2937	0.2362	0.2437
	(0.49,0.10)	0.9283	0.9557	0.2610	0.3061	0.2034	0.1679
	(0.50,0.05)	0.9254	0.9472	0.2626	0.3054	0.1557	0.1048
(50,50)	(0.05,0.50)	0.9453	0.9490	0.2012	0.2201	0.1159	0.0842
	(0.10,0.49)	0.9471	0.9499	0.2014	0.2198	0.1630	0.1431
	(0.15,0.45)	0.9481	0.9514	0.1904	0.2099	0.1876	0.1970
	(0.28,0.28)	0.9628	0.9643	0.1582	0.1784	0.1582	0.1784
	(0.45,0.15)	0.9476	0.9520	0.1901	0.2097	0.1874	0.1974
	(0.49,0.10)	0.9473	0.9488	0.2014	0.2198	0.1630	0.1426
	(0.50,0.05)	0.9457	0.9468	0.2026	0.2201	0.1150	0.0840
(100,100)	(0.05,0.50)	0.9552	0.9420	0.1424	0.1484	0.0786	0.0619
	(0.10,0.49)	0.9535	0.9432	0.1417	0.1478	0.1239	0.1166
	(0.15,0.45)	0.9556	0.9479	0.1341	0.1404	0.1340	0.1399
	(0.28,0.28)	0.9549	0.9555	0.1111	0.1178	0.1111	0.1178
	(0.45,0.15)	0.9528	0.9513	0.1341	0.1405	0.1340	0.1400
	(0.49,0.10)	0.9555	0.9480	0.1415	0.1482	0.1244	0.1162
	(0.50,0.05)	0.9572	0.9404	0.1425	0.1486	0.0784	0.0614
(200,200)	(0.05,0.50)	0.9500	0.9400	0.1005	0.1024	0.0540	0.0466
	(0.10,0.49)	0.9523	0.9410	0.1000	0.1019	0.0946	0.0935
	(0.15,0.45)	0.9512	0.9410	0.0946	0.0968	0.0946	0.0968
	(0.28,0.28)	0.9496	0.9499	0.0783	0.0806	0.0783	0.0806
	(0.45,0.15)	0.9494	0.9412	0.0946	0.0966	0.0946	0.0966
	(0.49,0.10)	0.9510	0.9400	0.1001	0.1020	0.0945	0.0934
	(0.50,0.05)	0.9550	0.9380	0.1006	0.1024	0.0538	0.0462

Table 4. The ratio of expected length of the 95% confidence intervals for the difference of coefficients of variation, when

(n,m)	(τ_1,τ_2)	$\frac{E(CI_{ds})}{E(CI_{sDR})}$	$\frac{E(CI_{dw})}{E(CI_{wDR})}$	$\frac{E(CI_{sDR})}{E(CI_{wDR})}$
(10,10)	(0.05,0.50)	1.4761	5.4174	1.7307
	(0.10,0.49)	1.3131	3.5116	1.2126
	(0.15,0.45)	1.1405	2.3096	0.8788
	(0.28,0.28)	1.0002	1.2164	0.4913
	(0.45,0.15)	1.1422	2.3340	0.8874
	(0.49,0.10)	1.3070	3.4972	1.2133
	(0.50,0.05)	1.4813	5.4499	1.7350
(30,30)	(0.05,0.50)	1.6813	2.8960	1.4810
	(0.10,0.49)	1.2872	1.8312	1.2127
	(0.15,0.45)	1.0471	1.2082	0.9699
	(0.28,0.28)	1.0000	1.0000	0.8117
	(0.45,0.15)	1.0451	1.2052	0.9694
	(0.49,0.10)	1.2830	1.8239	1.2118
	(0.50,0.05)	1.6863	2.9146	1.4862
(50,50)	(0.05,0.50)	1.7360	2.6140	1.3765
	(0.10,0.49)	1.2356	1.5360	1.1391
	(0.15,0.45)	1.0149	1.0656	0.9523
	(0.28,0.28)	1.0000	1.0000	0.8865
	(0.45,0.15)	1.0143	1.0619	0.9491
	(0.49,0.10)	1.2354	1.5414	1.1431
	(0.50,0.05)	1.7617	2.6188	1.3683
(100,100)	(0.05,0.50)	1.8122	2.3974	1.2695
	(0.10,0.49)	1.1430	1.2676	1.0629
	(0.15,0.45)	1.0008	1.0036	0.9576
	(0.28,0.28)	1.0000	1.0000	0.9429
	(0.45,0.15)	1.0011	1.0037	0.9570
	(0.49,0.10)	1.1379	1.2754	1.0704
	(0.50,0.05)	1.8188	2.4202	1.2762
(200,200)	(0.05,0.50)	1.8629	2.1974	1.1582
	(0.10,0.49)	1.0573	1.0898	1.0119
	(0.15,0.45)	1.0000	1.0000	0.9772
	(0.28,0.28)	1.0000	1.0000	0.9713
	(0.45,0.15)	1.0000	1.0000	0.9789
	(0.49,0.10)	1.0592	1.0921	1.0119
	(0.50,0.05)	1.8712	2.2165	1.1637

et al. (2015). From Table 6, the ratios of expected length between CI_{rs} and CI_{sRR}, and CI_{rw} and CI_{wRR}, respectively, are greater than one, especially when τ_1 and τ_2 are close to the boundary of parameter spaces. Moreover, the expected lengths of CI_{sRR} are slightly smaller than those of CI_{wRR}.

10. Real Data Examples

Example 1. The data of monthly rainfall (mm) are used to compute the confidence intervals for τ. From the report of the Hydrology Irrigation Center for the central region of Thailand (2015), 60 observations in September (between 1955-2014), which is the month with the lowest rainfall, are selected. The statistics are reported as follows.

Before computing the confidence intervals, the Anderson-Darling test is used to test the distribution of these data. It was found that the data of rainfall in September fit the gamma distribution with the test statistic of 0.45 and the p-value of 0.25. The average rainfall is $\bar{x} = 238.02$, $\sum_{i=1}^{60} \ln x_i = 324.40$, and the maximum likelihood estimator for τ is $\hat{\tau} = 0.36$. Suppose that the bounded interval $0.30 < \tau < 0.50$ is

Table 5. The coverage probabilities and expected lengths of the 95% confidence intervals for the ratio of coefficients of variation, when $0.05 < \tau_1, \tau_2 < 0.51$

(n,m)	(τ_1,τ_2)	Coverage probability		Expected length			
		CI_{rs}, CI_{sRR}	CI_{rw}, CI_{wRR}	CI_{rs}	CI_{rw}	CI_{sRR}	CI_{wRR}
(10,10)	(0.05,0.50)	0.9960	0.9969	0.2720	0.2803	0.2094	0.2168
	(0.10,0.49)	0.9967	0.9971	0.5557	0.5725	0.5268	0.5423
	(0.15,0.45)	0.9963	0.9969	0.9151	0.9427	0.9081	0.9351
	(0.28,0.28)	0.9965	0.9968	2.7670	2.8507	2.7660	2.8496
	(0.45,0.15)	0.9959	0.9968	8.4905	8.7474	7.4015	7.5410
	(0.49,0.10)	0.9971	0.9974	13.8598	14.2791	8.1611	8.2310
	(0.50,0.05)	0.9965	0.9972	28.2733	29.1287	6.5183	6.6051
(30,30)	(0.05,0.50)	0.9640	0.9663	0.0834	0.0842	0.0518	0.0525
	(0.10,0.49)	0.9632	0.9652	0.1703	0.1721	0.1699	0.1717
	(0.15,0.45)	0.9682	0.9699	0.2794	0.2824	0.2794	0.2824
	(0.28,0.28)	0.9683	0.9700	0.8496	0.8585	0.8496	0.8585
	(0.45,0.15)	0.9688	0.9707	2.5769	2.6041	2.5769	2.6041
	(0.49,0.10)	0.9692	0.9709	4.2272	4.2718	4.1886	4.2310
	(0.50,0.05)	0.9642	0.9659	8.6765	8.7680	3.2560	3.2846
(50,50)	(0.05,0.50)	0.9593	0.9603	0.0598	0.0602	0.0351	0.0354
	(0.10,0.49)	0.9578	0.9589	0.1221	0.1229	0.1221	0.1229
	(0.15,0.45)	0.9605	0.9617	0.2004	0.2017	0.2004	0.2017
	(0.28,0.28)	0.9628	0.9643	0.6118	0.6158	0.6118	0.6158
	(0.45,0.15)	0.9611	0.9626	1.8581	1.8700	1.8581	1.8700
	(0.49,0.10)	0.9568	0.9572	3.0540	3.0736	3.0528	3.0724
	(0.50,0.05)	0.9578	0.9588	6.2299	6.2700	2.5382	2.5528
(100,100)	(0.05,0.50)	0.9538	0.9543	0.0402	0.0404	0.0225	0.0226
	(0.10,0.49)	0.9495	0.9504	0.0822	0.0825	0.0822	0.0825
	(0.15,0.45)	0.9502	0.9508	0.1347	0.1351	0.1347	0.1351
	(0.28,0.28)	0.9549	0.9555	0.4108	0.4121	0.4108	0.4121
	(0.45,0.15)	0.9521	0.9528	1.2497	1.2538	1.2497	1.2538
	(0.49,0.10)	0.9543	0.9548	2.0488	2.0554	2.0488	2.0554
	(0.50,0.05)	0.9551	0.9563	4.1894	4.2031	1.8230	1.8285
(200,200)	(0.05,0.50)	0.9456	0.9461	0.0278	0.0279	0.0152	0.0152
	(0.10,0.49)	0.9470	0.9472	0.0568	0.0569	0.0568	0.0569
	(0.15,0.45)	0.9479	0.9483	0.0932	0.0934	0.0932	0.0934
	(0.28,0.28)	0.9496	0.9499	0.2837	0.2842	0.2837	0.2842
	(0.45,0.15)	0.9442	0.9448	0.8638	0.8652	0.8638	0.8652
	(0.49,0.10)	0.9444	0.9448	1.4172	1.4196	1.4172	1.4196
	(0.50,0.05)	0.9476	0.9481	2.8923	2.8971	1.3145	1.3166

considered. The 95% confidence intervals are calculated and presented in Table 7. Like the simulation in the previous section, CI_{sR} has a shorter length than CI_{wR}. The length of CI_s is greater than that of CI_{sR} and CI_{wR}. Meanwhile, the length of CI_w is not different from that of CI_{wR}, but greater than that of CI_{sR}.

Example 2. We use the data provided by Proschan (1963) to compute the confidence intervals for ψ and η. These data present the time (hours) of successive failures of the air conditioning systems of two jet airplanes, number 7912 and 7909. Using the Anderson-Darling test, the data fit the gamma distribution. For the first plane, 30 samples are observed with the sample mean $\bar{x} = 59.60$ and the sample coefficient of variation $\hat{\tau}_1 = 1.21$. For 29 samples from the second plane, the sample mean and sample coefficient of variation are $\bar{y} = 83.52$ and $\hat{\tau}_2 = 0.81$, respectively.

Suppose that the coefficients of variation are bounded, $0.80 < \tau_1, \tau_2 < 1.30$. The 95% confidence intervals and the lengths for ψ are presented in Table 7. It can be seen that the lengths of CI_{sDR} and CI_{wDR} are shorter than those of CI_{ds} and CI_{dw}. That means the proposed confidence intervals are better than the existing intervals. In addition, these data are

Table 6. The ratio of expected length of the 95% confidence intervals for the ratio of coefficients of variation, when $0.05 < \tau_1, \tau_2 < 0.51$

(n,m)	(τ_1,τ_2)	$\frac{E(CI_{rs})}{E(CI_{sRR})}$	$\frac{E(CI_{rw})}{E(CI_{wRR})}$	$\frac{E(CI_{sRR})}{E(CI_{wRR})}$
(10,10)	(0.05,0.50)	1.2990	1.2926	0.9659
	(0.10,0.49)	1.0548	1.0557	0.9714
	(0.15,0.45)	1.0076	1.0081	0.9711
	(0.28,0.28)	1.0003	1.0004	0.9707
	(0.45,0.15)	1.1471	1.1600	0.9815
	(0.49,0.10)	1.6983	1.7348	0.9915
	(0.50,0.05)	4.3375	4.4100	0.9869
(30,30)	(0.05,0.50)	1.6081	1.6060	0.9882
	(0.10,0.49)	1.0023	1.0024	0.9896
	(0.15,0.45)	1.0000	1.0000	0.9896
	(0.28,0.28)	1.0000	1.0000	0.9896
	(0.45,0.15)	1.0000	1.0000	0.9896
	(0.49,0.10)	1.0092	1.0096	0.9900
	(0.50,0.05)	2.6648	2.6694	0.9913
(50,50)	(0.05,0.50)	1.7018	1.7006	0.9929
	(0.10,0.49)	1.0001	1.0001	0.9936
	(0.15,0.45)	1.0000	1.0000	0.9936
	(0.28,0.28)	1.0000	1.0000	0.9936
	(0.45,0.15)	1.0000	1.0000	0.9936
	(0.49,0.10)	1.0004	1.0004	0.9936
	(0.50,0.05)	2.4544	2.4561	0.9943
(100,100)	(0.05,0.50)	1.7851	1.7846	0.9965
	(0.10,0.49)	1.0000	1.0000	0.9968
	(0.15,0.45)	1.0000	1.0000	0.9968
	(0.28,0.28)	1.0000	1.0000	0.9968
	(0.45,0.15)	1.0000	1.0000	0.9968
	(0.49,0.10)	1.0000	1.0000	0.9968
	(0.50,0.05)	2.2982	2.2986	0.9969
(200,200)	(0.05,0.50)	1.8319	1.8317	0.9983
	(0.10,0.49)	1.0000	1.0000	0.9984
	(0.15,0.45)	1.0000	1.0000	0.9984
	(0.28,0.28)	1.0000	1.0000	0.9984
	(0.45,0.15)	1.0000	1.0000	0.9984

used to conduct the 95% confidence intervals for η. We also note that these results support the simulation studies in the previous section.

11. Conclusions

The aim of this paper was twofold. The first aim was to propose new confidence intervals for the coefficient of variation with bounded parameter space in the gamma distribution. These estimators were extended from the confidence intervals based on the score and Wald intervals. The second aim was to propose new confidence intervals for the difference and the ratio of coefficients of variation with bounded parameter spaces in two gamma distributions. The results obtained from the simulations indicated that the proposed confidence intervals had the same coverage probabilities as their existing confidence intervals. However, all proposed confidence intervals performed better than the compared estimators in terms of expected length, especially when the coefficients of variation were close to the boundary. That means the proposed confidence intervals more accurately estimate the true parameter.

Therefore, it should be noted that when the coefficient of variation is known to be bounded, the confidence interval utilized the information about the restriction of the parameter is used. Since our estimators outperform and are also easy to

Table 7. The results of the 95% confidence intervals and the lengths of interval for the data examples

Results	The existing confidence interval		The proposed confidence interval	
CIs for τ	CI_s	CI_w	CI_{sR}	CI_{wR}
Interval	[0.29,0.42]	[0.31,0.45]	[0.30,0.42]	[0.31,0.45]
Length	0.13	0.14	0.12	0.14
CIs for ψ	CI_{ds}	CI_{dw}	CI_{sDR}	CI_{wDR}
Interval	[-0.01,0.77]	[-0.02,0.93]	[-0.01,0.45]	[-0.02,0.45]
Length	0.78	0.95	0.46	0.47
CIs for η	CI_{rs}	CI_{rw}	CI_{sRR}	CI_{wRR}
Interval	[0.99,2.25]	[0.98,2.25]	[0.99,1.63]	[0.98,1.63]
Length	1.26	1.27	0.64	0.65

compute using the explicit formulas, those estimators can be used as the confidence intervals for functions of coefficients of variation with bounded parameter spaces in the gamma distributions.

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