ORIGINAL ARTICLE

Roots of Matrices

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Abstract

Yuttanan, B. and Nilrat, C.
Roots of Matrices
Songklanakarin J. Sci. Technol., 2005, 27(3) : 659-665

A matrix S is said to be an n^{th} root of a matrix A if $S^n = A$, where n is a positive integer greater than or equal to 2. If there is no such matrix for any integer $n \geq 2$, A is called a rootless matrix. After investigating the properties of these matrices, we conclude that we always find an n^{th} root of a non-singular matrix and a diagonalizable matrix for any positive integer n. On the other hand, we find some matrix having an n^{th} root for some positive integer n. We call it p-nilpotent matrix.

Key words : roots of matrices, rootless matrix, nilpotent matrix, non-singular matrix, diagonalizable matrix

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Received, 18 May 2004 Accepted, 11 October 2004

บทคัดย่อ

บุญรอด ยุทธานันท์ และ ช่อฟ้า นิลรัตน์

รากของเมทริกซ์

ว. สงขลานครินทร์ วทท. 2548 27(3) : 659-665

เรากล่าวว่า S เป็นรากที่ n ของเมทริกซ์ A ถ้า S^n = A เมื่อ n เป็นจำนวนเต็มบวกที่มากกว่าหรือเท่ากับ 2 ถ้าไม่มีเมทริกซ์ S และจำนวนเต็มบวก n ดังกล่าว เราเรียก A ว่าเมทริกซ์ที่ไม่มีราก หลังจากทำการตรวจสอบสมบัติของเมทริกซ์เหล่านี้ พบว่าสำหรับทุกจำนวนเต็มบวก n เราสามารถหารากที่ n ของเมทริกซ์ไม่เอกฐานและเมทริกซ์ที่คล้ายกับเมทริกซ์ทแยงมุมได้เสมอ นอกจากนี้เรายังพบว่ามีบางเมทริกซ์ที่เราสามารถหารากที่ n ได้เพียงบางจำนวนเต็มบวก n เท่านั้น เมทริกซ์ดังกล่าวคือ เมทริกซ์พี-นิลโพเทนต์

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An $m\times m$ matrix A is called *nilpotent* if $A^r = 0$ for some positive integer $r \geq 2$. Yood (2002) showed that any nilpotent $m\times m$ matrix A such that A^{m-1} 0 is rootless. Such a matrix is called *principal nilpotent*. After we finished reading this article, we raised the question of which matrices always have an n^{th} root for any positive integer n and which have an n^{th} root only for some positive integer n. In this paper, we give the answer for these questions.

1. Roots of non-singular matrices

In this section, we prove that every non-singular matrix has an n^{th} root for any positive integer. Before discussing on a non-singular matrix, we start with a property of upper triangular matrices.

Lemma 1.1 *If* $A = [a_{ij}]_{m\times m}$ *is an upper triangular matrix, then so is* $A^n = [\alpha_{ij}]_{m\times m}$ *and*

$$\alpha_{ij} = \begin{cases} 0 & \text{if } i > j, \\ a_{ii}^n & \text{if } i = j, \\ \sum\limits_{i\le k_1\le \ldots \le k_{n-1}\le j} a_{ik_1}a_{k_1k_2}\ldots a_{k_{n-1}j} & \text{if } i < j. \end{cases}$$

Proof. We give a proof by mathematical induction. For $n = 2$, we have

$$A^2 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{pmatrix}^2 = \begin{pmatrix} a_{11}^2 & a_{11}a_{12}+a_{12}a_{22} & \cdots & a_{11}a_{1m}+a_{12}a_{2m}+\ldots+a_{1m}a_{mm} \\ 0 & a_{22}^2 & \cdots & a_{22}a_{2m}+a_{23}a_{3m}+\ldots+a_{2m}a_{mm} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm}^2 \end{pmatrix} =: \left[\alpha_{ij}\right].$$

Apparently, A^2 is an upper triangular matrix such that for each i, j, $1 \leq i, j \leq m$,

$$\alpha_{ij} = \begin{cases} 0 & \text{if } i > j, \\ a_{ii}^2 & \text{if } i = j, \\ \sum\limits_{i\le k_1\le j} a_{ik_1}a_{k_1j} & \text{if } i < j. \end{cases}$$

Now, we assume that $A^k = [\alpha_{ij}]$ where

$$\alpha_{ij} = \begin{cases} 0 & \text{if } i > j, \\ a_{ii}^k & \text{if } i = j, \\ \sum\limits_{i \le k_1 \le \ldots \le k_{k-1} \le j} a_{ik_1} a_{k_1 k_2} \ldots a_{k_{k-1} j} & \text{if } i < j. \end{cases}$$

Then

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ 0 & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_{mm} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{11}a_{11} & \alpha_{11}a_{12} + \alpha_{12}a_{22} & \cdots & \alpha_{11}a_{1m} + \alpha_{12}a_{2m} + \ldots + \alpha_{1m}a_{mm} \\ 0 & \alpha_{22}a_{22} & \cdots & \alpha_{22}a_{2m} + \alpha_{23}a_{3m} + \ldots + \alpha_{2m}a_{mm} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_{mm}a_{mm} \end{pmatrix} \\ &= :\left[\alpha'_{ij}\right]. \end{aligned}$$

It is clear that $\alpha'_{ij} = 0$ if $i > j$. For each integer i, $1 \le i \le m$, $\alpha'_{ii} = \alpha_{ii}a_{ii} = a_{ii}^k a_{ii} = a_{ii}^{k+1}$. We also obtain

$$\alpha'_{ij} = \sum_{k=i}^{j} \alpha_{ik} a_{kj} = \sum_{k=i}^{j} \left(\sum_{i \le k_1 \le \ldots \le k_{k-1} \le k} a_{ik_1} a_{k_1k_2} \ldots a_{k_{k-1}k} \right) a_{kj} = \sum_{i \le k_1 \le \ldots \le k_k \le j} a_{ik_1} a_{k_1k_2} \ldots a_{k_k j}$$

for all integers i and j, $1 \le i < j \le m$. □

Theorem 1.2 *Let A be an m×m complex matrix. If A is non-singular, then A always has an n^{th} root for any positive integer n.*

Proof. Let A be non-singular. By Schur's theorem (Strang, 1988), there exists a non-singular matrix S such that $A = SBS^{-1}$ where B is upper triangular. Let $B = \left[b_{ij}\right]_{m \times m}$. We have $\det(B) \ne 0$; that is, $b_{ii} \ne 0$ for $i = 1, 2, \ldots, m$. Let b_{ii}^* be any n^{th} root of b_{ii}. If $b_{ii} = b_{i'i'}$, we let $b_{ii}^* = b_{i'i'}^*$. We define $C = \left[c_{ij}\right]_{m \times m}$ as follows. For each i, $c_{ii} = b_{ii}^*$. For $i > j$, let $c_{ij} = 0$. For $j = i+1$, let $c_{ij} = b_{ij} \Big/ \sum_{p=0}^{n-1} c_{ii}^{n-1-p} c_{jj}^p$. For $j = i+k$, where $2 \le k \le m-i$, let $c_{ij} = (b_{ij} - R_{ij}) \Big/ \sum_{p=0}^{n-1} c_{ii}^{n-1-p} c_{jj}^p$, and R_{ij} be the sum of the products $c_{ik_1} c_{k_1k_2} \cdots c_{k_{n-1}j}$, where the sum is taken over integers $k_1, k_2, \ldots, k_{n-1}$ such that $i \le k_1 \le \ldots \le k_{n-1} \le j$ and none of the term in the products contains c_{ij}.

Since $b_{ii} \neq 0$ for *i* = *1, 2, ..., m*, we have $c_{ii} \neq 0$ for each *i* and $c_{i,i}^{n-1} + c_{i,i}^{n-2}c_{i+k,i+k} + \ldots + c_{i+k,i+k}^{n-1} \neq 0$ for $1 \leq k \leq m-i$. This guarantees that $c_{i,i+k}$ is well-defined. We claim that $C^n = B$.

Let $C^n = \left[\gamma_{ij}\right]_{m\times m}$. By Lemma 1.1, we have

$$\gamma_{ij} = \begin{cases} 0 & \text{if } i > j, \\ c_{ii}^n & \text{if } i = j, \\ \sum\limits_{i \leq k_1 \leq \ldots \leq k_{n-1} \leq i+k} c_{i,k_1} c_{k_1,k_2} \ldots c_{k_{n-1},i+k} & \text{if } j = i+k, k = 1,2,\ldots,m-i. \end{cases}$$

If $i = j, \gamma_{ij} = c_{ii}^n = (b_{ii}^*)^n = b_{ii}$.

If $j = i+1$, we have

$$\begin{aligned} \gamma_{i,i+1} &= \sum_{i \leq k_1 \leq \ldots \leq k_{n-1} \leq i+1} c_{i,k_1} c_{k_1,k_2} \ldots c_{k_{n-1},i+1} \\ &= c_{i,i+1}(c_{i,i}^{n-1} + c_{i,i}^{n-2}c_{i+1,i+1} + \ldots + c_{i+1,i+1}^{n-1}) \\ &= \frac{b_{i,i+1}(c_{i,i}^{n-1} + c_{i,i}^{n-2}c_{i+1,i+1} + \ldots + c_{i+1,i+1}^{n-1})}{c_{i,i}^{n-1} + c_{i,i}^{n-2}c_{i+1,i+1} + \ldots + c_{i+1,i+1}^{n-1}} \\ &= b_{i,i+1}. \end{aligned}$$

If $j = i+k$, when *k* = 2, 3, ..., *m-i*, we have

$$\begin{aligned} \gamma_{i,i+k} &= \sum_{i \leq k_1 \leq \ldots \leq k_{n-1} \leq i+k} c_{i,k_1} c_{k_1,k_2} \ldots c_{k_{n-1},i+k} \\ &= c_{i,i+k}(c_{i,i}^{n-1} + c_{i,i}^{n-2}c_{i+k,i+k} + \ldots + c_{i+k,i+k}^{n-1}) + R_{i,i+k} \\ &= \frac{(b_{i,i+k} - R_{i,i+k})(c_{i,i}^{n-1} + c_{i,i}^{n-2}c_{i+k,i+k} + \ldots + c_{i+k,i+k}^{n-1})}{c_{i,i}^{n-1} + c_{i,i}^{n-2}c_{i+k,i+k} + \ldots + c_{i+k,i+k}^{n-1}} + R_{i,i+k} \\ &= b_{i,i+k}. \end{aligned}$$

Then we obtain $\left[\gamma_{ij}\right] = \left[b_{ij}\right]$. Therefore $A = (SCS^{-1})^n$. □

We illustrate the procedure in Theorem 1.2 by the following example. Let

$$B = \begin{pmatrix} 8 & -12 & 7 & -8 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & -28 \\ 0 & 0 & 0 & 8 \end{pmatrix}.$$

A third root of *B* is a matrix $C = \left[c_{ij}\right]$ where $c_{ij} = 0$, if $i > j$ and $c_{11} = 2$, $c_{22} = -1$, $c_{33} = 1$, $c_{44} = 2$,

$$c_{12} = b_{12} \big/ \left[c_{11}^2 + c_{11}c_{22} + c_{22}^2\right] = (-12) \big/ \left[2^2 + (2)(-1) + (-1)^2\right] = -4,$$

$$c_{23} = b_{23} \big/ \left[c_{22}^2 + c_{22}c_{33} + c_{33}^2\right] = (0) \big/ \left[(-1)^2 + (-1)(1) + 1^2\right] = 0,$$

$$c_{34}=b_{34}\Big/\left[c_{33}^2+c_{33}c_{44}+c_{44}^2\right]=(-28)\Big/\left[1^2+(1)(2)+2^2\right]=-4,$$

$$c_{13}=\left[b_{13}-\{c_{11}c_{12}c_{23}+c_{12}c_{22}c_{23}+c_{12}c_{23}c_{33}\}\right]\Big/\left[c_{11}^2+c_{11}c_{33}+c_{33}^2\right]$$
$$=\left[7-\{(2)(-4)(0)+(-4)(-1)(0)+(-4)(0)(1)\}\right]\Big/\left[2^2+(2)(1)+1^2\right]=1,$$

$$c_{24}=\left[b_{24}-\{c_{22}c_{23}c_{34}+c_{23}c_{33}c_{34}+c_{23}c_{34}c_{44}\}\right]\Big/\left[c_{22}^2+c_{22}c_{44}+c_{44}^2\right]$$
$$=\left[6-\{(-1)(0)(-4)+(0)(1)(-4)+(0)(-4)(2)\}\right]\Big/\left[(-1)^2+(-1)(2)+2^2\right]=2,$$

$$c_{14}=\left[b_{14}-\{c_{11}c_{12}c_{24}+c_{11}c_{13}c_{34}+c_{12}c_{22}c_{24}+c_{12}c_{23}c_{34}+c_{12}c_{24}c_{44}+c_{13}c_{33}c_{34}+c_{13}c_{34}c_{44}\}\right]\Big/\left[c_{11}^2+c_{11}c_{44}+c_{44}^2\right]$$
$$=\left[-8-\{(2)(-4)(2)+(2)(1)(-4)+(-4)(-1)(2)+(-4)(0)(-4)+(-4)(2)(2)+(1)(1)(-4)+\right.$$
$$\left.(1)(-4)(2)\}\right]\Big/\left[2^2+(2)(2)+2^2\right]=3.$$

That is $\begin{pmatrix}2 & -4 & 1 & 3\\ 0 & -1 & 0 & 2\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 2\end{pmatrix}$ is a third root of B.

Some singular matrices also have an n^{th} root such as

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}^n.$$

Moreover, we have a singular matrix $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ as a rootless matrix (Yood, 2002).

Corollary 1.3 *If all eigenvalues of A are not zero, then A has an n^{th} root.*
Proof. Since *A* has all non-zero eigenvalues, *A* is a non-singular matrix. □

Note. If only one eigenvalue of *A* is zero, in Theorem 1.2, we have $b_{ii} = 0$ for only one value of *i*. That means we still have $c_{i,i}^{n-1}+c_{i,i}^{n-2}c_{i+k,i+k}+\ldots+c_{i+k,i+k}^{n-1}\neq 0$. Then we can say that "A matrix with only one zero eigenvalue always has an n^{th} root".

2. Roots of diagonalizable matrices

In this section, we consider an n^{th} root of a diagonalizable matrix.

Theorem 2.1 *Let A be an $m\times m$ complex matrix. If A is diagonalizable, then A has an n^{th} root, for any positive integer n.*
Proof. Let *A* be a diagonalizable matrix, i.e., there exists a non-singular matrix *S* such that $A = SDS^{-1}$ where $D=\left[d_{ij}\right]_{m\times m}$ is a diagonal matrix.

Let $D^{1/n}=\left[d_{ij}^{1/n}\right]_{m\times m}$, where $d_{ij}^{1/n}$ is an n^{th} root of d_{ij}. So $A = S(D^{1/n})^nS^{-1} = (SD^{1/n}S^{-1})^n$. Therefore an n^{th} root of *A* exists. □

However, we have some non-diagonalizable matrices having an n^{th} root, for example,

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

has an n^{th} root because it is a non-singular matrix. Moreover, we see that diagonalizable matrices and non-singular matrices are not the only matrices which have an n^{th} root, since

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{n} & \frac{1}{n} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}^n, n \geq 2.$$

There are some further questions the reader might like to consider. For instance, what is a necessary and sufficient condition for a matrix to have an n^{th} root?

As an immediate consequence of the above theorem, we can conclude that a matrix A with all distinct eigenvalues has an n^{th} root. On the other hand, a real symmetric matrix also has an n^{th} root for any positive integer n, as well as a complex Hermitian matrix and a normal matrix.

3. Roots of *p*-nilpotent matrices

In the previous two sections, we considered matrices whose n^{th} root always exists for any positive integer n. In this section, we consider some kind of matrices which has an n^{th} root for just some value of n.

An $m\times m$ matrix A is called *p-nilpotent* if A is a nilpotent matrix but not principal nilpotent and p is the least positive integer such that $A^p = 0$ but A^{p-1} 0. Before discussing on p-nilpotent matrices, we first give the following lemma.

Lemma 3.1 *Let A be an $m\times m$ complex matrix. If $A^k = 0$ for some $k \geq 2$, then $A^m = 0$.*

Proof. If $2 \leq k \leq m$, then we are done. Now we suppose $k > m$. By Schur's theorem (Strang, 1988), there exists a non-singular matrix S such that $A = SBS^{-1}$ where B is upper triangular. Since $A^k = 0$, we have $B^k = 0$.

Let $B = \left[b_{ij}\right]_{m\times m}$ and $B^k = \left[\beta_{ij}\right]_{m\times m}$. For $1 \leq i \leq m$, we have $\beta_{ii} = b_{ii}^k$, so $b_{ii} = 0$. Then B is strictly upper triangular. It was proved by Yood (2002) that $B^m = 0$. Therefore $A^m = 0$. □

Theorem 3.2 *Let A be an $m\times m$ p-nilpotent matrix. If an n^{th} root of A exists, then $n \leq m - p + 1$.*

Proof. The proof is by contradiction. Suppose that $A = S^r$, for $r \geq m - p + 2$. Then $S^{rp} = A^p = 0$ so that S is an $m\times m$ nilpotent matrix. By Lemma 3.1, the m^{th} power of S is zero. Therefore, $S^k = 0$ for all positive integers $k \geq m$. But we also have $S^{r(p-1)} = A^{p-1}$ 0. Now $p \geq 2$, hence, $2r - 2 \leq rp - p$, so that $r + p - 2 \leq$ rp - r. Since $r \geq m - p + 2$, $m \leq r + p - 2 \leq rp - r$. Therefore $S^{rp-r} = 0$ or $A^{p-1} = 0$, which is contrary to the hypotheses on A. Hence, if an n^{th} root of p-nilpotent matrix exists, then $n \leq m - p + 1$. □

Let A be a 2-nilpotent matrix of size 3×3, i.e., $A^2 = 0$. By Schur's theorem, A is of the form SBS^{-1} where S is non-singular and B is upper triangular. Hence $B^2 = 0$. This implies $B^3 = 0$. By Yood (2002), B is a strictly upper triangular matrix. It is possible to classify B which is not principal nilpotent as five different types:

$$\begin{pmatrix}0&a&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&a\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&a\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&a&b\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&b\\0&0&a\\0&0&0\end{pmatrix},$$

where a, b 0.

We observe that

$$\begin{pmatrix}0&a&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&a\\0&0&0\\0&1&0\end{pmatrix}^2,$$

$$\begin{pmatrix}0&0&0\\0&0&a\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&a\\1&0&0\\0&0&0\end{pmatrix}^2,$$

$$\begin{pmatrix}0&0&a\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a&0\\0&0&1\\0&0&0\end{pmatrix}^2,$$

$$\begin{pmatrix}0&a&b\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&-a&0\\0&-1&-\frac{b}{a}\\0&\frac{a}{b}&1\end{pmatrix}^2,$$

$$\begin{pmatrix}0&0&b\\0&0&a\\0&0&0\end{pmatrix}=\begin{pmatrix}1&-\frac{b}{a}&0\\\frac{a}{b}&-1&-a\\0&0&0\end{pmatrix}^2.$$

Then we see that all five types of B has a square root, say T. Therefore $A = ST^2S^{-1} = (STS^{-1})^2$. This shows that a square root of any 2-nilpotent matrix of size 3×3 always exists.

Conclusion and Discussion

According to this article, we obtain a formula for calculating an n^{th} root of a matrix which is non-singular or diagonalizable.

However, being non-singular or diagonalizable are not necessary for matrices to have n^{th} roots. The reader may try to find other properties of his own. In addition, a matrix having an n^{th} root for some positive integer n is not only a p-nilpotent matrix.

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