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The degree and the order of polynomials in the ring $R\left[F_A^1\right]$

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Abstract

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In this research, we generalize some properties of the degree and the order of polynomials in the ring $R[x]$.

Key words : polynomials, degree, order

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บทคัดย่อ

รณสรรพ์ ชินรัมย์
ระดับขั้นและอันดับของพหุนามในริง $R\left[F_A^1\right]$
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ในงานวิจัยนี้ เราขยายสมบัติบางอย่างของระดับขั้นและอันดับของพหุนามในริง $R[x]$.

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Let R be a ring. f is said to be a *polynomial in x with coefficients in R* if f is the form of sum

$$f=\sum_{n=0}^{\infty}a_n x^n \text{ where } a_n \in R \text{ for all } n\in \mathbf{N}\cup\{0\}$$

such that $a_n = 0$ for all but a finite number of indices n.

Let $R[x]$ be the set of all polynomials in x with coefficients in R.

For any $f=\sum_{n=0}^{\infty}a_n x^n$ and $g=\sum_{n=0}^{\infty}b_n x^n$, define binary operations $+$ and $\cdot$ on $R[x]$ by

$$f+g=\sum_{n=0}^{\infty}(a_n+b_n)x^n$$

and

$$f\cdot g=\sum_{n=0}^{\infty}\sum_{i=0}^{n}a_i b_{n-i}x^n=\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_n b_m x^{n+m}.$$

Then $R[x]$ is a ring under these two binary operations + and $\cdot$. The ring $R[x]$ is called the *ring of polynomials in x with coefficients in R* or the *polynomial ring* (see Hungerford, 1980).

Let $f \in R[x]$ where $f \neq 0$. The *degree* of f is $\max\{n \mid a_n \neq 0\}$. The degree of f is denoted by deg f. A polynomial $f\in R[x]$ such that deg $f = n$ will always be written in the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where $a_i \in R$ and $a_n \neq 0$. The next theorem shows some properties of the degree of polynomials in the ring $R[x]$.

Theorem 1.1 (See Hungerford, 1980).

Let R be a ring and $f, g \in R[x] \setminus \{0\}$. The following statements are true.

(i) $fg = 0$ or deg $(fg) \leq$ deg f + deg g.

(ii) If R is an integral domain, then deg (fg) = deg f + deg g.

(iii) $f + g = 0$ or deg $(f + g) \leq \max\{\deg f, \deg g\}$.

(iv) If deg $f \neq$ deg g, then deg$(f + g) = \max\{\deg f(x), \deg g(x)\}$.

Let $f \in R[x]$ where $f \neq 0$. The order of f is $\min\{n \mid a_n \neq 0\}$ and the order of 0 is ∞. The order of f is denoted by ord f (see Grillet, 1999). The next theorem shows some properties of the order of the polynomials in the ring $R[x]$.

Theorem 1.2 (see Grillet, 1999).

Let R be a ring and f, g $\in R[x] \setminus \{0\}$. The following statements are true.

(i) $fg = 0$ or ord $(fg) \geq$ ord f + ord g.

(ii) If R is an integral domain, then ord (fg) = ord f + ord g.

(iii) $f + g = 0$ or ord $(f + g) \geq \min\{\text{ord } f, \text{ord } g\}$.

(iv) If ord $f \neq$ ord g, then ord $(f + g) = \min\{\text{ord } f, \text{ord } g\}$.

In this paper, we generalize Theorem 1.1 and Theorem 1.2.

The semigroup F_A^1.

Let A be any nonempty set and

$$F_A = \{a_1a_2...a_m \mid m \in \mathbf{N}, a_i \in A \text{ for all } i \in \{1,...,m\}\}.$$

For $a_1a_2...a_m, b_1b_2...b_n \in F_A$ define a binary operation on F_A by

$(a_1a_2...a_m)(b_1b_2...b_n) = a_1a_2...a_mb_1b_2...b_n.$ (2.1)

It is easy to prove that F_A is a semigroup under this binary operation. F_A is said to be a *free semigroup* generated by the set A and A is called a *generating set* of F_A (see Howie, 1975).

Next, let 1 be a new element and

$F_A^1 = F_A \cup \{1\}.$

Define a binary operation on F_A^1 by

$x1 = x = 1x$ for all $x \in F_A^1$ and $1 \cdot 1 = 1$

and if $x, y \in F_A^1$ then xy satisfies (2.1).

We can easy to see that F_A^1 is a semigroup under the above binary operation.

Example 2.1. Let $A = \{x\}$. Then

1. By the definition of free semigroups F_A, we have

$F_A = \{x, x^2, x^3, ...\} = \{x^n \mid n \in \mathbf{N}\}$

and x^ix^j for all $i, j \in \mathbf{N}$.

2. By the definition of the semigroups F_A^1, we have

$F_A^1 = \{1, x, x^2, x^3, ...\} = \{x^n \mid n \in \mathbf{N} \cup \{0\}\}$

where $1 = x^0$ and $1x^i = x^i = x^i1$ for all $i \in \mathbf{N}$, $1 \cdot 1 = 1$ and $x^ix^j = x^{i+j}$ for all $i, j \in \mathbf{N}$.

Theorem 2.1.

If $|A| > 1$, then F_A^1 is not a commutative semigroup.

Proof. Assume that $|A| > 1$. Then there exist $x, y \in A$ such that $x \neq y$. Then $xy, yx \in F_A^1$ and xy yx. Thus F_A^1 is not a commutative semigroup.

Let A be a nonempty set. For $s = a_1a_2...a_n \in F_A$ where $a_i \in A$ for all $i \in \{1, 2, ..., n\}$, the *length* of s is n and the *length* of 1 is 0. For $s \in F_A^1$, the length of s is denoted by $L(s)$.

Example 2.2.

Let $A = \{a, b\}$, Then $1, aba, a^2bab^3 \in F_A^1$. By definition of the length of element in F_A^1, we have that

$L(1) = 0$, $L(aba) = 3$ and $L(a^2bab^3) = 7$.

The next theorem shows some properties of the length of elements in F_A^1.

Theorem 2.2.

Let A be any nonempty set. For $x, y \in F_A^1$, we have

$L(xy) = L(x) + L(y)$.

Proof. Let A be a nonempty set and $x, y \in F_A^1$. Then $x = 1$ or $x \in F_A$ and $y = 1$ or $y \in F_A$.

Case 1 : $x = 1$. Then $L(x) = 0$. Thus
$L(xy) = L(y) = 0 + L(y) = L(x) + L(y)$.

Case 2 : $x \in F_A$. Then $x = a_1a_2...a_n$ for some $a_1,a_2,...a_n \in A$.

Case 2.1: $y = 1$. Then $L(y) = 0$. Thus
$L(xy) = L(x) = L(x) + 0 = L(x) + L(y)$.

Case 2.2: $y \in F_A$. Then there exist $b_1,b_2,...b_m \in A$ such that $y = b_1b_2...b_m$. We have that

$xy = a_1a_2...a_nb_1b_2...b_m$,

so

$L(xy) = m + n = L(x) + L(y)$.

Therefore, $L(xy) = L(x) + L(y)$ for all $x, y \in F_A^1$.

The ring $R[S]$.

Let R be a ring and S a semigroup. f is said to be a *polynomial on S with coefficients in R* if f is the form of finite sums $f = \sum_{s \in S} a_s s$ where $s \in S$ and $a_s \in S$.

Let $R[S]$ be the set of all polynomials on S with coefficients in R. For any $f = \sum_{s \in S} a_s s$ and $g = \sum_{s \in S} b_s s$, define binary operations + and $\cdot$ on $R[S]$ by

$$f + g = \sum_{s \in S} a_s s + \sum_{s \in S} b_s s = \sum_{s \in S} (a_s + b_s)s$$

and

$$f \cdot g = (\sum_{s \in S} a_s s)(\sum_{s' \in S} b_{s'} s') = \sum_{s \in S}\sum_{s' \in S} (a_s b_{s'})(ss').$$

Then $R[S]$ is a ring under these binary operations + and $\cdot$ (see Zhang, Chen and Li, 1996).

If $A = \{x\}$, by Example 2.1, we have known that $F_A^1 = \{1, x, x^2, ...\}$. By the definition of the rings $R[S]$, it is easy to see that $R[F_A^1] = R[x]$. Therefore, the ring $R[S]$ is a generalization of the ring $R[x]$.

The degree of polynomials in the ring $R[F_A^1]$.

Let A be a nonempty set and R a ring. Let $f \in R[F_A^1] \setminus \{0\}$. such that $f = \sum_{s \in F_A^1} a_s s$. The *degree* of f is $\max\{L(s) \mid a_s \quad 0\}$. For $f \in R[F_A^1]$, the degree of f is denoted by deg f.

Example 4.1.

Let $A = \{a, b\}$ and $\mathbf{R}$ be the set of all real numbers. Let $f = 2ab + 3a^3 + 4b^2a^2 \in \mathbf{R}[F_A^1]$. By the definition of the degree of elements in $\mathbf{R}[F_A^1]$, it is easy to see that deg $f = 4$.

In the remainder of this section, let A be a nonempty set and R a ring.

Theorem 4.1.

Let $f, g \in R[F_A^1]$. If $f \neq 0$ and $g \neq 0$, then $fg = 0$ or deg $(fg) \leq$ deg f + deg g.

Proof. Let $f = \sum_{s \in S} a_s s$ and $g = \sum_{s \in S} b_s s$ be any two elements in $R[F_A^1]$ such that $f \neq 0$ and $g \neq 0$. We have $fg = \sum_{s \in S} \sum_{s' \in S} (a_s b_{s'})(ss')$. Assume $fg \neq 0$. Thus

$$
\begin{aligned}
\deg(fg) &= \max\{L(ss') \mid a_s b_{s'} \neq 0\} \\
&= \max\{L(s) + L(s') \mid a_s b_{s'} \neq 0\} \\
&\quad \text{by Theorem 2.2} \\
&\leq \max\{L(s) + L(s') \mid a_s \neq 0 \text{ and } b_{s'} \quad 0\} \\
&\leq \max\{L(s) \mid a_s \neq 0\} + \max\{L(s') \mid b_{s'} \neq 0\} \\
&= \deg f + \deg g.
\end{aligned}
$$

Theorem 4.2.

Let R be an integral domain and $f, g \in R[F_A^1]$. If $f \neq 0$ and $g \neq 0$, then deg (fg) = deg f + deg g.

Proof. Let $f = \sum_{s \in S} a_s s$ and $g = \sum_{s \in S} b_s s$ be any two elements in R[F_A^1] such that $f \neq 0$ and $g \neq 0$. Assume deg $f = n$ and deg $g = m$. Let $u \in \{s \in F_A^1 \mid L(s) = n$ and $a_s \neq 0\}$ and $v \in \{s \in F_A^1 \mid L(s) = m$ and $b_s \neq 0\}$. So $a_u \neq 0$ and $b_v \neq 0$. Since R is an integral domain, $a_u b_v \neq 0$. By properties of u and v, we have

$$f = a_u u + \sum_{\substack{s \in S \\ L(s) \leq n \\ s \neq u}} a_s s \quad \text{and} \quad g = g_v v + \sum_{\substack{s \in S \\ L(s) \leq n \\ s \neq v}} b_s s.$$

Thus $fg = a_u b_v uv + \sum_{\substack{s \in S \\ L(ss') \leq m+n \\ ss' \neq uv}} b_s s.$

From Theorem 2.2, we have known that $L(uv) = L(u) + L(v) = m + n$. Hence,

deg $(fg) = m + n =$ deg f + deg g,

as required.

Theorem 4.3.

Let $f, g \in R[F_A^1]$. If $f \neq 0$ and $g \neq 0$, then $f + g = 0$ or deg $(f + g) \leq \max\{\deg f, \deg g\}$.

Proof. Let $f = \sum_{s \in S} a_s s$ and $g = \sum_{s \in S} b_s s$ be any two elements in $R[F_A^1]$ such that $f \neq 0$ and $g \neq 0$.

Let deg $f = n$ and deg $g = m$.

Case 1: $m > n$. We have that

$$
\begin{aligned}
f + g &= \sum_{s \in S} a_s s + \sum_{s \in S} b_s s \\
&= \sum_{\substack{s \in S \\ L(s) \leq n}} (a_s + b_s) s + \sum_{\substack{s \in S \\ L(s) > n}} b_s s
\end{aligned}
$$

so

$$
\begin{aligned}
\deg(f + g) &= \max\{L(s) \mid b_s \neq 0 \text{ and } L(s) > n\} \\
&= m \\
&= \max\{m, n\} \\
&= \max\{\deg f, \deg g\}.
\end{aligned}
$$

Case 2: $n > m$. We have that

$$f + g = \sum_{s \in S} a_s s + \sum_{s \in S} b_s s$$

$$= \sum_{\substack{s\in S\\ L(s)\le m}}(a_s+b_s)s+\sum_{\substack{s\in S\\ L(s)>m}}a_s s$$

so

$$\begin{aligned}\deg(f+g) &= \max\{L(s) \mid a_s \neq 0 \text{ and } L(s) > \mathrm{m}\}\\ &= n\\ &= \max\{m, n\}\\ &= \max\{\deg f, \deg g\}.\end{aligned}$$

Case 3: $m = n$. We have that

$$f+g = \sum_{s\in S}a_s s+\sum_{s\in S}b_s s$$

Case 3.1 : $g = -f$. So $f + g = 0$.

Case 3.2 : $g \neq -f$. So $f + g \neq 0$. Then

$$\begin{aligned}\deg(f+g) &= \max\{L(s) \mid a_s + b_s \neq 0\}\\ &\le \max\{\max\{L(s) \mid a_s \neq 0 \text{ or } b_s \neq 0\}\}\\ &\le \max\{\max\{L(s) \mid a_s \neq 0\}, \max\{L(s) \mid b_s \neq 0\}\}\\ &= \max\{\deg f, \deg g\}.\end{aligned}$$

Therefore, $f + g = 0$ or $\deg(f + g) \le \max\{\deg f, \deg g\}$, as required.

Corollary 4.4.

Let $f,g \in R[F_A^1]$ such that $f \neq 0$ and $g \neq 0$. If $\deg f \neq \deg g$, then $\deg(f + g) = \max\{\deg f, \deg g\}$.

Proof. By the proof of Case 1 and Case 2 of Theorem 4.3.

The order of polynomials in the ring $R[F_A^1]$.

Let A be a nonempty set and R a ring. For $f \in R[F_A^1]$ such that $f = \sum_{s\in F_A^1} a_s s$. If $f \neq 0$, the *order* of f is $\min\{L(s) \mid a_s \neq 0\}$ and the order of 0 is ∞. For $f \in R[F_A^1]$, the order of f is denoted by ord f.

Example 5.1.

Let $A = \{a, b\}$ and $\mathbf{R}$ be the set of all real numbers. Let $f = 2ab + 3a^3 + 4b^2a^2 \in \mathbf{R}[F_A^1]$. By the definition of the order of elements in $\mathbf{R}[F_A^1]$, it is easy to see that ord $f = 2$.

In the remainder of this section, let A be a nonempty set and R a ring.

Theorem 5.1.

Let $f,g \in R[F_A^1]$. If $f \neq 0$ and $g \neq 0$, then $fg = 0$ or $\operatorname{ord}(fg) \ge \operatorname{ord} f + \operatorname{ord} g$.

Proof. Let $f = \sum_{s\in S} a_s s$ and $g = \sum_{s\in S} b_s s$ be any two elements in $R[F_A^1]$ such that $f \neq 0$ and $g \neq 0$. We have $fg = \sum_{s\in S}\sum_{s'\in S}(a_s b_{s'})(ss')$. Assume that $fg \neq 0$. Then we have that

$$\begin{aligned}\operatorname{ord}(fg) &= \min\{L(ss') \mid a_s b_{s'} \neq 0\}\\ &= \min\{L(s) + L(s') \mid a_s b_{s'} \neq 0\}\\ &\quad \text{by Theorem 2.2}\\ &\ge \min\{L(s) + L(s') \mid a_s \neq 0 \text{ and } b_{s'} \neq 0\}\\ &\ge \min\{L(s) \mid a_s \neq 0\} + \min\{L(s') \mid b_{s'} \neq 0\}\\ &= \operatorname{ord} f + \operatorname{ord} g.\end{aligned}$$

Hence, $\operatorname{ord}(fg) > \operatorname{ord} f + \operatorname{ord} g$.

Theorem 5.2.

Let R be an integral domain and $f,g \in R[F_A^1]$. If $f \neq 0$ and $g \neq 0$, then $\operatorname{ord}(fg) = \operatorname{ord} f + \operatorname{ord} g$.

Proof. Let $f = \sum_{s\in S} a_s s$ and $g = \sum_{s\in S} b_s s$ be any two elements in $R[F_A^1]$ such that f 0 and $g \neq 0$. Assume ord $f = n$ and ord $g = m$. Let $u \in \{s \in F_A^1 \mid L(s) = n$ and $a_s \neq 0\}$ and $v \in \{s \in \ \mid L(s) = m$ and $b_s \neq 0\}$. So $a_u \neq 0$ and $b_v \neq 0$. Since R is an integral domain, $a_u b_v \neq 0$. By properties of u and v, we have

$$f = a_u u + \sum_{\substack{s\in S\\ L(s)\ge n\\ s\neq u}} a_s s \quad\text{and}\quad g = b_v v + \sum_{\substack{s\in S\\ L(s)\ge m\\ s\neq v}} b_s s.$$

Thus $$fg = a_u b_v uv + \sum_{\substack{s\in S\\ L(ss')\ge m+n\\ ss'\neq uv}} a_s b_{s'} ss'.$$

By Theorem 2.2, we have $L(uv) = L(u) + L(v) = m+n$. Hence, $\operatorname{ord}(fg) = m + n = \operatorname{ord} f + \operatorname{ord} g$, as required.

Theorem 5.3.

Let $f,g \in R[F_A^1]$. If $f \neq 0$ and $g \neq 0$, then $f + g = 0$ or $\operatorname{ord}(f + g) \ge \min\{\operatorname{ord} f, \operatorname{ord} g\}$.

Proof. Let $f = \sum_{s \in S} a_s s$ and $g = \sum_{s \in S} b_s s$ be any two elements in $R[F_A^1]$ such that $f \neq 0$ and $g \neq 0$.

Let ord $f = n$ and ord $g = m$.

Case 1: $m > n$. We have that

$$f + g = \sum_{s \in S} a_s s + \sum_{s \in S} b_s s$$
$$= \sum_{\substack{s \in S \\ L(s) \geq m}} (a_s + b_s)s + \sum_{\substack{s \in S \\ L(s) < m}} a_s s$$

so

$$\begin{aligned} \text{ord } (f + g) &= \min\{L(s) \mid a_s \neq 0 \text{ and } L(s) < m\} \\ &= n \\ &= \min\{m, n\} \\ &= \min\{\text{ord } f, \text{ord } g\}. \end{aligned}$$

Case 2: $n > m$. We have that

$$f + g = \sum_{s \in S} a_s s + \sum_{s \in S} b_s s$$
$$= \sum_{\substack{s \in S \\ L(s) \geq n}} (a_s + b_s)s + \sum_{\substack{s \in S \\ L(s) < n}} b_s s$$

so

$$\begin{aligned} \text{ord } (f + g) &= \min\{L(s) \mid b_s \neq 0 \text{ and } L(s) < n\} \\ &= m \\ &= \min\{m, n\} \\ &= \min\{\text{ord } f, \text{ord } g\}. \end{aligned}$$

Case 3: $m = n$. We have that

$$f + g = \sum_{s \in S} (a_s + b_s)s$$

Case 3.1: $g = -f$. So $f + g = 0$.

Case 3.2: $g \neq -f$. So $f + g \neq 0$. Then

$$\begin{aligned} \text{ord } (f + g) &= \min\{L(s) \mid a_s + b_s \neq 0\} \\ &\geq \min\{\min\{L(s) \mid a_s \neq 0 \text{ or } b_s \neq 0\}\} \\ &\geq \min\{\min\{L(s) \mid a_s \neq 0\}, \min\{L(s) \mid b_s \neq 0\}\} \\ &= \min\{\text{ord } f, \text{ord } g\} \end{aligned}$$

Therefore $f + g = 0$ or ord $(f + g) \geq \min\{$ord f, ord $g\}$, as required.

Corollary 5.4.

Let $f, g \in R[F_A^1]$ such that $f \neq 0$ and $g \neq 0$. If ord $f \neq$ ord g, then ord$(f + g) = \min\{$ord f, ord $g\}$.

Proof. By the proof of Case 1 and Case 2 of Theorem 5.3.

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References

Grillet, P.A. 1999. Algebra, John wiley & Sons, Inc., New York.

Howie, J.M. 1975. An introduction to semigroup theory, Academic Press, London.

Hungerford, T.W. 1980. Algebra, Springer-Verlag, New York.

Zhang, M.C., Chen Y.Q. and Li, Y.H. 1996. The intersection property of quasi-ideals in rings and semigroup rings , SEA Bull. Math., 20 : 117-122.