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Original Article

Hajos stable graphs

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Abstract

Let G_1 and G_2 be two undirected graphs, (v w) be an edge of G_1, and (x y) be an edge of G_2. Hajos construction forms a new graph H, that combines the two graphs by identifying vertices v and x into a single vertex, removing the two edges (v w) and (x y), and adding a new edge (w y). In this paper we introduce a new kind of graph called the Hajos stable graph, where the stable property is defined using the domination number of graphs G_1 and G_2. We have obtained a necessary and sufficient condition for graphs G_1 and G_2 to be Hajos stable.

Keywords: domination number, Hajos construction, Hajos graph, stable graph

1. Introduction

Gyorgy Hajos was a Hungarian mathematician who worked in group theory, graph theory, and geometry. In graph theory, a branch of mathematics, the Hajos construction is an operation on graphs named after Gyorgy Hajos. Catlin has provided a conjecture for Hajos graph coloring (1979). Brown (1990) has proved that the Hajos construction of two amenable k–critical graphs need be amenable for any $k \geq 5$ (1990). An analogue of Hajos theorem for the circular chromatic number was proved by Zhu (2001).

Kral studied an analogue of Hajos theorem for list coloring. Also one of the operations of Hajos sum was introduced by Kral (2004). Hajos join construction was introduced by Liu (2006).

A graph is total domination dot stable if dotting any pair of adjacent vertices leaves the total domination number unchanged (Rickett *et al.*, 2011). Desormeaux *et al.*, studied total domination stable graphs upon edge removal (2011).

Graph operations produce new graphs from the initial one. Binary operations are in general tough, since they involve more than one graph. Researchers have attempted to determine the domination number of graphs resulting by binary operations. The Hajos construction is a binary operation involving three operations of edge removal, edge addition, and vertex merging.

We have attempted to find the domination number of a Hajos graph, with an additional constraint that the domination number of the resulting graph is equal to the sum of the domination numbers of the graphs from which the Hajos graph was constructed. In the process of determining this we could find the conditions under which the domination number of the Hajos graph is less than (greater than) the sum of the domination numbers of the graphs from which the Hajos graph was constructed.

2. Materials and Methods

We consider only simple connected undirected graphs G = (V, E). The open neighborhood of vertex $v \in V$ (G) is denoted by N (v) = {u ∈ V (G) | (u v) ∈ E (G)} while its closed neighborhood is the set N [v] = N (v) ∪ {v}. We indicate that u is adjacent to v by writing $u \perp v$.

A set of vertices D, in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set. The cardinality of any minimum dominating set for G is called the domination number of G and it is denoted by γ (G). γ-set denotes a dominating set for G with minimum cardinality.

A vertex v is said to be good if there is a γ-set of G containing v. If there is no γ-set of G containing v, then v is

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said to be bad vertex. A vertex v is said to be a down vertex if γ (G – u) < γ (G), level vertex if γ (G – u) = γ (G), and up vertex if γ (G – u) > γ (G). A vertex v is said to be selfish in the γ-set D if v is needed only to dominate itself. A vertex in V – D is k–dominated if it is dominated by at least k vertices in D that is $| N (v) \cap D | \geq k$. The private neighborhood of v $\in$ D is denoted by pn [v, D] and is defined by pn [v, D] = N (v) – N (D − {v}). For details on domination we refer to (Haynes *et al.*, 1998).

2.1 Hajos construction

Let G_1 and G_2 be two graphs, (u_1 v_1) be an edge of G_1, and (u_2 v_2) be an edge of G_2. Then the Hajos construction forms a new graph H that combines the two graphs by merging the vertices u_1 and u_2 into a single vertex u_{12}, removing the two edges (u_1 v_1) and (u_2 v_2), and adding a new edge (v_1 v_2) (Brown *et al.*, 1990).

Example

In Figure 1, H_1, H_2, and H_3 are the Hajos graphs obtained from G_1 and G_2 using the edge pairs {(u_4 u_6), (v_2 v_3)}, {(u_1 u_2), (v_4 v_5)}, and {(u_1 u_2), (v_4 v_7)}, respectively, and γ (H_1) > γ (G_1) + γ (G_2), γ (H_2) < γ (G_1) + γ (G_2), and γ (H_3) = γ (G_1) + γ (G_2), respectively.

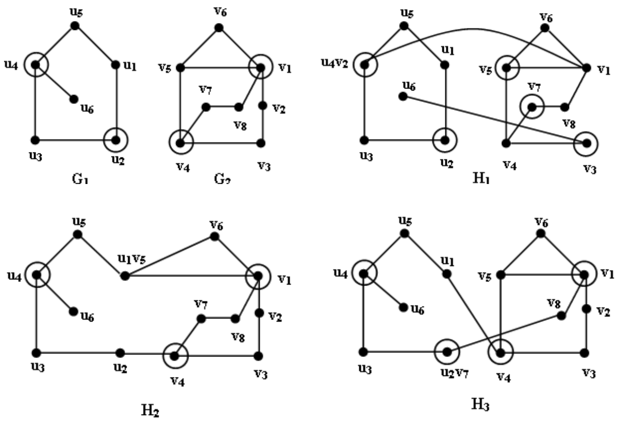

Figure 1. H_1, H_2 and H_3 are the Hajos graph obtained from G_1 and G_2 using the edge pairs {(u_4 u_6), (v_2 v_3)}, {(u_1 u_2), (v_4 v_5)}, {(u_1 u_2), (v_4 v_7)} respectively. γ (H_1) > γ (G_1) + γ (G_2), γ (H_2) < γ (G_1) + γ (G_2), γ (H_3) = γ (G_1) + γ (G_2).

From this example we understand that, when we construct the Hajos graph, the domination number of H may be greater than, less than or equal to γ (G_1) + γ (G_2). Based on this we define a new type of graph called the Hajos stable graph. However, before defining it we note that, when we pick a pair of edges say (u_1 v_1) $\in$ V (G_1), (u_2 v_2) $\in$ V (G_2), the following four Hajos graphs are constructed.

1. H_1 - merging vertices u_1u_2, adding an edge between v_1, v_2 and deleting edges (u_1 v_1), (u_2 v_2).
2. H_2 - merging vertices u_1v_2, adding an edge between v_1, u_2 and deleting edges (u_1 v_1), (u_2 v_2).
3. H_3 - merging vertices v_1u_2, adding an edge between u_1, v_2 and deleting edges (u_1 v_1), (u_2 v_2).
4. H_4 - merging vertices v_1v_2, adding an edge between u_1, u_2 and deleting edges (u_1 v_1), (u_2 v_2).

So, for every pair of edges, four Hajos graphs are possible. We consider only those graphs for which the Hajos graph is always connected.

2.2 Hajos stable graphs

Let G_1 and G_2 be any two graphs. Let E (G_1) = {e_{11}, e_{12},…, e_{1p}} and E (G_2) = {e_{21}, e_{22},…, e_{2q}}. Let M = E (G_1) × E (G_2) = {(e_{1i} e_{2j}) | e_{1i} $\in$ E (G_1), e_{2j} $\in$ E (G_2)}, that is M is the cartesian product between sets E (G_1) and E (G_2). Let |M| = k. Let H_1, H_2,…, H_{4k} be the Hajos graphs generated by applying Hajos construction 4k times. If $\gamma(H_i) = \gamma(G_1) + \gamma(G_2)$, for all i = 1, 2,…, 4k, then G_1 and G_2 are said to be *Hajos stable graphs*.

3. Results and Discussion

In this section we provide a necessary and sufficient condition for graphs G_1 and G_2 to be Hajos stable. We discuss properties satisfied by Hajos stable graphs.

Type I operation

Split a vertex u_{12} into u_1 and u_2 so that every edge that was incident on u_{12} is now incident either on u_1 or u_2.

Let G_1 and G_2 be any two graphs as seen in Figure 2 (a). Let H be the Hajos graph created by choosing arbitrary edges e_i = (u_i v_i) in G_i, i = 1, 2, as seen in Figure 2 (b). Let D = {u_{12}, a_1, b_2} be a γ-set for H. G_i–{e_i} are the graphs generated by applying Type I operation on u_{12} and removing edge (v_1 v_2) in H. We try to retain back the γ-set of H in G_i– {e_i} as far as possible. Since u_{12} $\in$ D, we shall include u_{12} either in G_1–{e_1} or G_2–{e_2}, say in G_1–{e_1} as seen in Figure 2 (c).

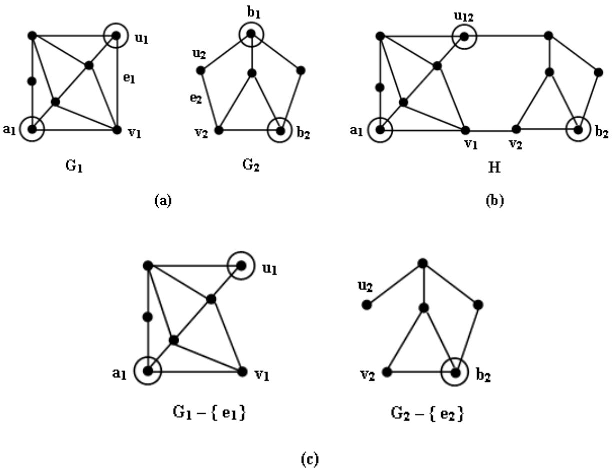

Figure 2. Sample illustration for γ-set retaining.

We shall follow this operation of splitting H into $G_i - \{e_i\}$ (using Type I operation and edge removal) and retaining back the γ-set of H in $G_i - \{e_i\}$ in proof discussions, when needed. So, in many discussions we will pick two graphs G_1 and G_2, H the Hajos graph created from G_1 and G_2, and D a γ-set for H. Whenever needed we split H into $G_i-\{e_i\}$. $D_i = D \cap V(G_i)$ (except for u_{12} included either in D_1 or D_2, if $u_{12} \in D$).

Notations

We shall use the following notations throughout the paper.

1. G_1 and G_2 are any two graphs.
2. H is the Hajos graph generated from G_1 and G_2.
3. $e_i = (u_i v_i)$, i = 1, 2 are any two arbitrary edges from G_1 and G_2 respectively.
4. While creating a Hajos graph using any $e_i = (u_i v_i) \in E(G_i)$, i = 1, 2, the vertex obtained by merging vertices u_1, u_2 is labeled as u_{12}.
5. Whenever $\gamma(H) = \gamma(G_1) + \gamma(G_2)$, let R be a γ-set for H. In this case let $A_i = G_i-\{e_i\}$. Let $X_i = R \cap V(G_i)$, $Y_i = X_i \cup \{u_i\}$, i = 1, 2.
6. Whenever $\gamma(H) < \gamma(G_1) + \gamma(G_2)$, let S be a γ-set for H. In this case, let $B_i = G_i-\{e_i\}$. Let $Z_i = S \cap V(G_i)$, $W_i = Z_i \cup \{u_i\}$, i = 1, 2.
7. Whenever $\gamma(H) > \gamma(G_1) + \gamma(G_2)$, let L be a γ-set for H. In this case, let $C_i = G_i-\{e_i\}$. Let $P_i = L \cap V(G_i)$, $Q_i = P_i \cup \{u_i\}$, i = 1, 2.
8. Throughout the paper when we discuss splitting the Hajos graph into two parts (either G_i or A_i or B_i or C_i), we retain back the γ-sets as they are in the Hajos graph H except for vertex u_{12}. Discussions related to u_{12} will vary. We shall explain the changes when required.

In the following theorems we prove that the domination number of the Hajos graph increases or decreases by atmost one.

Theorem 1

Let G_1 and G_2 be any two graphs and H be the Hajos graph. If $\gamma(H) > \gamma(G_1) + \gamma(G_2)$, then $\gamma(H) = \gamma(G_1) + \gamma(G_2) + 1$.

Proof

Let G_1 and G_2 be two graphs such that $\gamma(H) > \gamma(G_1) + \gamma(G_2)$. Let $e_1 = (u_1 v_1)$, $e_2 = (u_2 v_2)$ be any two arbitrary edges in G_1 and G_2 respectively.

1. Let $u_1, v_1 \in D_1$. We know that $\gamma(G_2) \geq 1$, say $u_2 \in D_2$. Then $D_1 \cup D_2 - \{u_1\} - \{u_2\} \cup \{u_{12}\}$ is a dominating set for H such that $\gamma(H) < \gamma(G_1) + \gamma(G_2)$. So, G_1 and G_2 are any two graphs such that there is no $u_i, v_i \in G_i$, $u_i \perp v_i$, $(u_i, v_i) \in D_i$, i = 1, 2. So, any γ-set for G_i is independent.
2. We consider the following cases.
   (a) $u_i, v_i \notin D_i$, i = 1, 2
   (b) $u_1 \notin D_1, v_1 \in D_1, u_2 \notin D_2, v_2 \notin D_2$
   (c) $u_1 \notin D_1, v_1 \notin D_1, u_2 \notin D_2, v_2 \in D_2$
   (d) $u_1 \in D_1, v_1 \notin D_1, u_2 \notin D_2, v_2 \in D_2$
   (e) $u_1 \notin D_1, v_1 \in D_1, u_2 \in D_2, v_2 \notin D_2$.

   In all these cases $D_1 \cup D_2$ or $D_1 \cup D_2 - \{u_i\} \cup \{u_{12}\}$, i =1, 2 is a dominating set for H, a contradiction to our assumption. So, we consider graphs G_i, which do not contain D_i as in (a) to (e).
3. $u_1 \in D_1, v_1 \notin D_1, u_2 \in D_2, v_2 \notin D_2$. $D_1 \cup D_2 - \{u_1\} - \{u_2\} \cup \{u_{12}\} \cup \{v_1\}$ is a dominating set for H such that $\gamma(H) \leq \gamma(G_1) + \gamma(G_2)$, a contradiction to our assumption. So, we consider graphs for which there are no γ-sets $u_1 \in D_1, v_1 \notin D_1, u_2 \in D_2, v_2 \notin D_2$.
4. We consider the following cases.
   i. If $u_1 \in D_1, v_1 \notin D_1, u_2 \notin D_2, v_2 \notin D_2$, $D_1 \cup D_2 - \{u_1\} \cup \{u_{12}\} \cup \{v_1\}$ is a dominating set for H.
   ii. If $u_1, v_1 \notin D_1, u_2 \in D_2, v_2 \notin D_2$, $D_1 \cup D_2 - \{u_2\} \cup \{u_{12}\} \cup \{v_1\}$ is a dominating set for H.
   iii. If $u_1 \notin D_1, v_1 \in D_1, u_2 \notin D_2, v_2 \in D_2$, $D_1 \cup D_2 \cup \{u_{12}\}$ is a dominating set for H.

In all the cases i, ii and iii, $\gamma(H) = \gamma(G_1) + \gamma(G_2) + 1$. From 1, 2, 3, and 4, we conclude that if $\gamma(H) > \gamma(G_1) + \gamma(G_2)$, then $\gamma(H) = \gamma(G_1) + \gamma(G_2) + 1$.

Theorem 2

Let G_1 and G_2 be any two graphs and H be the Hajos graph. If $\gamma(H) < \gamma(G_1) + \gamma(G_2)$, then $\gamma(H) = \gamma(G_1) + \gamma(G_2) - 1$.

Proof

Let G_1 and G_2 be two graphs such that $\gamma(H) < \gamma(G_1) + \gamma(G_2)$. Let $e_1 = (u_1 v_1)$, $e_2 = (u_2 v_2)$ be any two arbitrary edges in G_1 and G_2 respectively. If possible assume that $\gamma(H) = \gamma(G_1) + \gamma(G_2) - k$, $k \geq 2$.

1. $u_{12}, v_1, v_2 \notin D$.

There are some $x \in V(G_i)$, i = 1, 2 or (say $V(G_1)$) such that $x \perp u_{12}$. Z_1, W_2 are dominating sets for G_1, G_2, respectively, such that $|Z_1 + W_2| < |D_1 + D_2|$, is a contradiction to our assumption that D_1 and D_2 are γ-sets for G_1, G_2 respectively.

2. $u_{12} \in D$

Z_1, W_2 or W_1, Z_2 are dominating sets for G_1, G_2, respectively, such that $|Z_1 + W_2| = |W_1 + Z_2| < |D_1 + D_2|$, a contradiction to our assumption.

3. $v_1 \in D$

Z_1, W_2 are dominating sets for G_1, G_2, respectively, such that $|Z_1 + W_2| < |D_1 + D_2|$, a contradiction to our assumption. Similarly we get a contradiction when $v_2 \in D$.

4. $u_{12}, v_1 \in D$ or $u_{12}, v_2 \in D$

In both cases Z_1, W_2 or W_1, Z_2 are dominating sets for G_1, G_2, respectively, such that $|Z_1 + W_2| = |W_1 + Z_2| < |D_1 + D_2|$, a contradiction to our assumption.

5. $v_1, v_2 \in D$ or $u_{12}, v_1, v_2 \in D$

In both cases Z_1, Z_2 are dominating sets for G_1 and G_2, respectively, such that $| Z_1 + Z_2 | < | D_1 + D_2 |$, a contradiction to our assumption.

In all possible cases, we get a contradiction. So, we conclude that if H is the Hajos graph such that $\gamma (H) < \gamma (G_1) + \gamma (G_2)$, then $\gamma (H) = \gamma (G_1) + \gamma (G_2) - 1$.

From Theorem 1 and 2, we can decide that, if H is the Hajos graph, then

$\gamma (H) = \gamma (G_1) + \gamma (G_2) + 1$ or

$\gamma (H) = \gamma (G_1) + \gamma (G_2) - 1$ or

$\gamma (H) = \gamma (G_1) + \gamma (G_2)$.

Theorem 3

Let G_1 and G_2 be any two graphs. Let D_1 and D_2 be γ-sets for G_1 and G_2 respectively. Let H be the Hajos graph. Then $\gamma (H) < \gamma (G_1) + \gamma (G_2)$ if and only if either

1. there is some $(u_i\, v_i) \in D_i$ such that $u_i \perp v_i$, i = 1, 2, or
2. there is a selfish vertex in G_i, i = 1, 2, or
3. both G_1 and G_2 have 2–dominated vertices simultaneously together, or
4. if pn $[u_i, D_i] = v_i$ in G_i, then G_j has 2–dominated vertices, where i, j = 1, 2, $i \neq j$.

Proof

Assume that $\gamma (H) < \gamma (G_1) + \gamma (G_2)$. By Theorem 2, we know that $\gamma (H) = \gamma (G_1) + \gamma (G_2) - 1$. Let S be a γ-set for H. Split the Hajos graph H to obtain B_1 and B_2. Either $| S \cap V (G_1) | < | D_1 |$ or $| S \cap V (G_2) | < | D_2 |$ (if $u_{12} \in S$, then either u_1 or u_2 is considered in the intersection). Without loss of generality throughout the proof assume that $| S \cap V (G_1) | < | D_1 |$, that is $| S \cap V (G_1) | = | D_1 | - 1$. If possible assume that conditions 1, 2, 3, and 4 are not satisfied.

Case 1 $u_{12}, v_1, v_2 \notin S$

If there is some $x \in V (G_1)$ dominating u_{12} in H, then Z_1 itself is a dominating set for G_1 such that $| Z_1 | < | D_1 |$, which is a contradiction as D_1 is a γ-set for G_1; or W_1 is a γ-set for G_1 such that u_1 selfish, a contradiction to our assumption that condition 2 is not satisfied.

Case 2 $u_{12} \in S$

If $u_1 \in \{S \cap V (G_1)\}$, then W_1 is a γ-set for G_1 such that $| W_1 | < | D_1 |$, a contradiction to the assumption that D_1 is a γ-set for G_1. $Z_1 \cup \{u_1\}$, W_2 are γ-sets for G_1 and G_2, respectively, such that v_1, v_2 are 2–dominated, a contradiction to our assumption that condition 3 is not satisfied.

Case 3 $v_1 \in S$

As $| S \cap V (G_1) | < | D_1 |$, Z_1 itself is a γ-set for G_1 such that $| Z_1 | < | D_1 |$, which is a contradiction as D_1 is a γ-set for G_1.

Case 4 $v_2 \in S$

Since $u_{12} \notin S$, there is some $x \in V (G_1)$ or $x \in V (G_2)$ such that $x \perp u_{12}$, $x \in S$. If there is some $x \in V (G_1)$, $x \in S$ such that x dominates u_{12}, then $Z_1 \cup \{v_1\}$ is a γ-set for G_1 such that v_1 is selfish, which is a contraction to our assumption that condition 2 is not satisfied.

If there is some $x \in V (G_2)$, $x \in S$ such that x dominates u_{12}, then Z_2 is a γ-set for G_2 such that u_2 is 2–dominated (x, v_2 adjacent to u_2). W_1 is a γ-set for G_1 such that pn $[u_1, W_1] = \{v_1\}$ (if $v_1 \in$ pn $[v_2, S]$), which is a contraction to our assumption that condition 4 is not satisfied; or W_1 is a γ-set for G_1 such that u_1 selfish (if $v_1 \notin$ pn $[v_2, S]$), which is a contraction to our assumption that condition 2 is not satisfied.

Case 5 $u_{12}, v_1 \in S$

Either $u_1 \in \{S \cap V (G_1)\}$ or $u_2 \in \{S \cap V (G_2)\}$. In both cases, we get a contradiction to our assumption that condition 1 is not satisfied. The above discussion is also true if $u_{12}, v_2 \in S$.

Case 6 $v_1, v_2 \in S$

Z_1 is a γ-set for G_1 such that $| Z_1 | < | D_1 |$, which is a contradiction as D_1 is a γ-set for G_1.

Case 7 $u_{12}, v_1, v_2 \in S$

Proof is similar to case 5. Conversely assume that the conditions of the theorem are satisfied. If possible assume that $\gamma (H) \geq \gamma (G_1) + \gamma (G_2)$.

1. If there is some $u_1, v_1 \in D_1$ such that $u_1 \perp v_1$. We know that, $\gamma (G_2) \geq 1$. So there is some vertex say $u_2 \in D_2$. $D_4 = D_1 \cup D_2 - \{u_1\} - \{u_2\} \cup \{u_{12}\}$ is a dominating set for H such that $| D_4 | < | D_1 | + | D_2 |$, which is a contradiction to our assumption that $\gamma (H) \geq \gamma (G_1) + \gamma (G_2)$.
2. Let $v \in V (G_1)$ such that v is selfish. Let N (v) = $\{v_1, v_2,\ldots, v_k\}$. We know that every vertex in N (v) is 2–dominated. $D_1 - \{v\} \cup \{v_i\}$, i = 1, 2,…, k are γ-sets for G_1 such that $v \perp v_i$. By the proof in condition 1, we know that there is a dominating set for H such that $\gamma (H) < \gamma (G_1) + \gamma (G_2)$, which is a contradiction to our assumption $\gamma (H) \geq \gamma (G_1) + \gamma (G_2)$.
3. Let $v_i \in V (G_i)$ such that v_i are 2 dominated vertices, i = 1, 2. Let $u_i, x_i \in D_i$, $v_i \perp u_i, x_i$, i = 1, 2. Choose edges $(u_i\, v_i)$, i = 1, 2 for Hajos construction. $D_1 \cup D_2 - \{u_1\} - \{u_2\} \cup \{u_{12}\}$ is a dominating set for H such that $\gamma (H) < \gamma (G_1) + \gamma (G_2)$, which is a contradiction to our assumption $\gamma (H) \geq \gamma (G_1) + \gamma (G_2)$.
4. Let pn $[u_1, D_1] = v_1$ in G_1 and u_2 a 2–dominated vertex in G_2. Since u_2 is 2–dominated, there are at least two vertices, say $v_2, x \in D_2$ such that $v_2, x \perp u_2$. $D_1 \cup D_2 - \{u_1\}$ is a dominating set for H (in H, u_{ij} is dominated by x and v_2

dominates v_1) such that γ (H) < γ (G_1) + γ (G_2), which is a contradiction to our assumption that γ (H) ≥ γ (G_1) + γ (G_2).

So, if the hypothesis of the theorem is satisfied, it is not possible that γ (H) ≥ γ (G_1) + γ (G_2). So, we conclude that γ (H) < γ (G_1) + γ (G_2).

Theorem 4

Let G_1 and G_2 be any two graphs. Let D_1 and D_2 be γ-sets for G_1 and G_2 respectively. Let H be the Hajos graph. Then γ (H) > γ (G_1) + γ (G_2) if and only if either

1. if u_i is an up vertex, u_j, v_i, v_j are bad vertices, then
   a. v_i is not a 2–dominated vertex with respect to every D_i in G_i and
   b. v_j is not a good vertex in C_j – N [u_j] for all γ - sets D_3 for C_j – N [u_j] such that | D_3 | = | D_j |, where i, j = 1, 2, i ≠ j, or
2. if u_i are bad vertices, v_i are up vertices, then u_i ∈ pn [v_i, D_i] for all possible γ-sets in G_i, i = 1, 2.

Proof

Let γ (H) > γ (G_1) + γ (G_2). Assume that u_1 is an up vertex, u_2, v_1, v_2 are bad vertices. If possible assume that v_1 is a 2–dominated vertex with respect to every D_1 in G_1 or v_2 is a good vertex in C_2 – N [u_2] for all γ-sets D_3 for C_2 – N [u_2] such that | D_3 | = | D_2 |.

If v_1 is 2–dominated with respect to D_1, then D_1 ∪ D_2 itself is a dominating set for H, which is a contradiction to the assumption that γ (H) > γ (G_1) + γ (G_2). If v_2 is a good vertex with respect to D_3, then D_1 ∪ D_3 is a dominating set for H (v_2 dominates v_1 and u_{12} dominates N [u_2] in H) such that | D_1 ∪ D_3 | = | D_1 ∪ D_2 |, which is a contradiction to the assumption that γ (H) > γ (G_1) + γ (G_2). Hence v_1 is not a 2–dominated vertex with respect to some D_1 in G_1 and v_2 is not a good vertex in C_2 – N [u_2] for all γ-sets D_3 for C_2 – N [u_2] such that | D_3 | = | D_2 |.

Assume that u_i are bad vertices, v_i are up vertices with respect to D_i, i = 1, 2. If possible assume that either u_1 or u_2 is 2–dominated with respect to some γ-set for G_1 or G_2, respectively. Let u_1 be a 2–dominated vertex with respect to D_1. Let v_1, x ∈ D_1 such that u_1 ⊥ v_1, x. D_1 is a γ - set for G_1 and C_1. D_2 is dominating at least C_2 – u_2. D_1 ∪ D_2 is a dominating set for H (since x_1 dominates u_{12} in H), which is a contradiction to the assumption that γ (H) > γ (G_1) + γ (G_2). Hence u_i ∈ pn [v_i, D_i] for all possible γ-sets D_i in G_i, i = 1, 2.

Conversely assume that the conditions of the theorem are satisfied. If possible assume that γ (H) ≤ γ (G_1) + γ (G_2). Let γ (H) = γ (G_1) + γ (G_2). Assume that condition 1 of the theorem is satisfied. Let u_1 be an up vertex, u_2, v_1, and v_2 bad vertices. Also assume that v_1 is not a 2–dominated vertex with respect to every D_1 in G_1 or v_2 is not a good vertex in C_2 – N [u_2] for all γ-sets D_3 for C_2 – N [u_2] such that | D_3 | = | D_2 |.

Let D be a γ-set for H. Either v_1 ∈ D or there is some x ∈ V (H) such that x dominates v_1. If v_1 ∈ D, then v_1 ∈ D_1, implies v_1 is a good vertex with respect to G_1, a contradiction to our assumption that v_1 is a bad vertex. If there is some x ∈ V (H) such that x dominates v_1, x ∈ V (G_1), then v_1 is 2–dominated with respect to D_1 (since u_1 is an up vertex, u_1 ∈ D_1, therefore, x, u_1 ∈ D_1 and x, u_1 ⊥ v_1), which is a contradiction to our assumption that v_1 is not a 2–dominated vertex with respect to D_1. If v_2 dominates v_1 in H, then D_2 contains v_2, which is a contradiction to our assumption that v_2 is a bad vertex in G_2.

Assume that condition 2 of the theorem is satisfied. Either u_{12} ∈ D or there is some y ∈ V (H) such that y dominates u_{12}. If u_{12} ∈ D, then there is a γ-set D_1 for G_1 containing u_1 or D_2 for G_2 containing u_2, a contradiction to our assumption that u_1 and u_2 are bad vertices with respect to G_1 and G_2, respectively. If there is some y ∈ V (G_1) such that y ⊥ u_{12}, then D_1 is a γ-set for G_1 such that y, v_1 ∈ D_1, implies u_1 is 2–dominated with respect to G_1, which is a contradiction to our assumption that u_1 ∈ pn [v_1, D_1] for every possible γ-set D_1 with respect to G_1. A similar argument results in a contradiction, if y ∈ V (G_2). We conclude that γ (H) ≠ γ (G_1) + γ (G_2). So, if the hypothesis of the theorem is satisfied, it is not possible that γ (H) ≤ γ (G_1) + γ (G_2). Hence we conclude that γ (H) > γ (G_1) + γ (G_2).

3.1 Necessary and sufficient condition for H to be Hajos stable

Theorem 5

Let G_1 and G_2 be any two graphs. Let D_1 and D_2 be γ-sets for G_1 and G_2 respectively. Let H be the Hajos graph. Then γ (H) = γ (G_1) + γ (G_2) if and only if

1. there is no (u_i, v_i) ∈ D_i such that u_i ⊥ v_i, i = 1, 2.
2. there is no selfish vertex in G_i, i = 1, 2.
3. both G_1 and G_2 do not have 2–dominated vertices simultaneously together.
4. if pn [u_i, D_i] = v_i in G_i, then G_j has no 2–dominated vertices, where i, j = 1, 2, and i ≠ j.
5. if u_i is an upvertex, u_j, v_i, v_j are bad vertices, then either v_i is a 2–dominated vertex with respect to some D_i in G_i or v_j is a good vertex in A_j – N [u_j] for some γ-set D_3 for A_j – N [u_j] such that | D_3 | = | D_j |, where i, j = 1, 2, and i ≠ j.
6. if u_i are bad vertices, v_i are up vertices, then either u_1 or u_2 (but not both) is 2–dominated with respect to some D_i, i = 1, 2.

Proof

Assume that γ (H) = γ (G_1) + γ (G_2). If possible assume that the conditions of the theorem are not satisfied. As discussed in sufficient part of Theorem 3 and 4, we get a contradiction to our assumption that γ (H) = γ (G_1) + γ (G_2). Hence the conditions of theorem are satisfied. Conversely assume that the conditions of the theorem are satisfied. If possible assume that γ (H) ≠ γ (G_1) + γ (G_2). So, γ (H) < γ (G_1) + γ (G_2) or γ (H) > γ (G_1) + γ (G_2). By Theorem 3, we know that if γ (H) < γ (G_1) + γ (G_2), then at least one of the conditions of Theorem 3 should be satisfied. But by our assumption we know that the conditions of Theorem 3 are not satisfied. Similarly by Theorem 4, we can conclude that the conditions of Theorem 4 are not satisfied. This means that

γ (H) $\geq \gamma$ (G_1) + γ (G_2) if 1, 2, 3, 4 are satisfied.

γ (H) $\leq \gamma$ (G_1) + γ (G_2) if 5, 6 are satisfied.

If conditions 1, 2, 3, and 4 are satisfied, then by Theorem 4, we know that γ (H) is not greater than γ (G_1) + γ (G_2). If conditions 5 and 6 are satisfied, then by Theorem 3, we know that γ (H) is not less than γ (G_1) + γ (G_2). So, if conditions 1–6 are satisfied, then neither γ (H) < γ (G_1) + γ (G_2) nor γ (H) > γ (G_1) + γ (G_2). This would be a contradiction to our assumption that γ (H) $\neq \gamma$ (G_1) + γ (G_2) and implies γ (H) = γ (G_1) + γ (G_2).

4. Conclusions

Binary graph operations are always exciting to begin with, but tough due to the complexity of its construction. Hajos construction is one such kind. Any new graph will have special properties satisfied when related with graph parameters. In this paper we have related the domination number of the Hajos graph with the domination number of the original graphs from where it was generated.

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