Songklanakarin J. Sci. Technol.
40 (3), 506-521, May - Jun. 2018

http://www.sjst.psu.ac.th

Original Article

Corrected score estimators in linear multivariate multiple regression models with heteroscedastic measurement errors

Wannaporn Junthopas, Jirawan Jitthavech*, and Vichit Lorchirachoonkul

School of Applied Statistics, National Institute of Development Administration, Bang Kapi, Bangkok, 10240 Thailand

Received: 28 September 2016; Revised: 9 December 2016; Accepted: 6 February 2017

Abstract

In this study, the knowledge of estimation theory based on the corrected score (CS) approach is extended in a linear multivariate multiple regression model with heteroscedastic measurement errors (HMEs) and an unknown HME variance. The heteroscedasticity of the HME variance is assumed to be capable of being grouped into similar patterns and can be estimated by the pooled variance of the observations of the variable with HME in repeated measurements. The statistical properties of the proposed CS estimator are analytically investigated. The simulation results confirm the theoretical results. The proposed CS estimator outperforms the OLS estimator under all simulation conditions.

Keywords: corrected score, grouped heteroscedasticity, heteroscedastic measurement errors, repeated measurements

1. Introduction

A heteroscedastic measurement error (HME) model is a statistical model in which some variables are measured with errors, and the variances of the measurement errors change across observations. HME models have been widely applied in epidemiology, analytical chemistry, and botany (Cheng & Riu, 2006; de Castro *et al.*, 2008; Kulathinal *et al.*, 2002; Patriota *et al.*, 2009; Veenendaal *et al.*, 2008). In some environments, the precise measurement of a specific variable is impracticable or very expensive in terms of time and effort and, furthermore, the error variances of this variable across observations may not be static. HME can occur in either the dependent variables (Y) or independent variables (Z), or both, and consequently, the ordinary least squares (OLS) assumptions are violated. Note that the OLS estimators in the case of HMEs in Y are unbiased whereas those in the case of HMEs in Z are biased and also inconsistent. Obviously, the HME problems in Z are more complicated than those in Y (Gujarati, 2006).

In the case of either measurement error (ME) or HME models, the methods used to correct the bias of the estimators can be grouped into either functional modeling or structural modeling. Some methods based on functional modeling are regression calibration, simulation-extrapolation, conditional score, corrected score (CS), and instrumental

*Corresponding author
Email address: jirawan@as.nida.ac.th

variables. In linear functional modeling, estimators from these methods have been shown to be asymptotically consistent (Buzas *et al.*, 2003).

The CS method has been widely investigated in the literature (Chen *et al.*, 2015; de Castro *et al.*, 2006; Giménez & Bolfarine, 1997; Giménez & Galae, 2013; Giménez & Patat, 2005, 2014; Huang & Wang, 2001;) and deals with parameter estimation in the presence of the ME of an independent variable by determining the biased correction term to correct the biased score estimation function. Nakamura (1990) proposed CS functions for four different models: the generalized linear model, the normal regression model, the Poisson regression model, and the gamma regression model and, since then, the CS approach has come to prominence in the literature. Giménez and Bolfarine (1997) derived the asymptotic distribution of CS estimators in a simple linear regression and a comparative calibration model. Next, a number of methods have been used in comparisons between the four approaches for consistent estimators: sufficiency and conditional scores, maximum likelihood estimation, a CS function, and moment estimators. It has been empirically shown using simulation studies that, for small and moderate sample sizes (*n* = 30, 50), there is no one estimator more efficient than the others (Giménez & Bolfarine, 2000). The assumption of known ME variance or HME variance is commonly applied in the studies of parameter estimation in a model with only one independent variable (Chen *et al.*, 2015; de Castro *et al.*, 2006; Giménez & Galae, 2013; Huang & Wang, 2001). Giménez and Patat (2005, 2014) proposed a method for estimating the parameter in a comparative calibration model under the unknown ME variance condition.

In this study, the CS approach is extended to a linear multivariate multiple regression model subjected to an unknown HME variance. The paper is structured as follows. Section 2 presents a novel method of parameter estimation in a linear multivariate regression model subject to an unknown HME variance, and the properties of proposed CS estimators are also investigated. In Section 3, the analytical results are confirmed by a simulation study, and conclusions drawn from the research are presented in Section 4.

2. Research Methodology

2.1 Model

Consider a linear multivariate measurement error regression model in which the p correlated dependent variables, $Y_1, Y_2, ..., Y_p$, are explained by s independent variables, where the first s_1 independent variables $Z_1, Z_2, ..., Z_{s_1}$ are precisely observed and the last $(s - s_1)$ independent variables, $Z_{s_1+1}, Z_{s_1+2}, ..., Z_s$, are imprecisely observed via their corresponding surrogate variables, $X_{s_1+1}, X_{s_1+2}, ..., X_s$, with additive HME. In the j^{th} observation, only dependent variables and the surrogate variables are measured repeatedly r_j times, $j = 1, 2, ..., n$ where n is the number of observations. Let z_{qj} be the value of precisely observed independent variable Z_q, $q = 1, 2, ..., s_1$ at the j^{th} observation, y_{ijk} and x_{qjk} be the k^{th} repeated measurement at the j^{th} observation of Y_i and X_q, respectively, $i = 1, 2, ..., p$, $q = s_1 + 1, s_1 + 2, ..., s$, $j = 1, 2, ..., n$, $k = 1, 2, ..., r_j$. Subsequently, the linear multivariate measurement error regression model can be expressed as

$$\mathbf{Y} = \mathbf{ZB} + \mathbf{E} \quad (1)$$

$$\bar{\mathbf{x}}_q = \mathbf{z}_q + \bar{\mathbf{u}}_q \;,\; q = s_1 + 1, s_1 + 2, ..., s \,. \quad (2)$$

Denote $\mathbf{Y}$ as the $n \times p$ matrix of the average measurements of dependent variables with the i^{th} and j^{th} elements $\bar{y}_{ij.} = \sum_{k=1}^{r_j} y_{ijk} / r_j$, $i = 1, 2, ..., p$, $j = 1, 2, ..., n$; $\mathbf{Z}$ as the $n \times (s+1)$ matrix of S independent variables including a constant unit vector, $\mathbf{z}_0 = \mathbf{1}$; $\mathbf{B}$ as the $(s+1) \times p$ matrix of parameters; $\mathbf{E}$ as the $n \times p$ matrix of random errors with the i^{th} and j^{th} elements $\bar{\varepsilon}_{ij.} = \sum_{k=1}^{r_j} \varepsilon_{ijk} / r_j$, where ε_{ijk} is the mutually independent random error in the k^{th} repeated measurement of dependent variable Y_i at the j^{th} observation; $\bar{\mathbf{x}}_q$ as the $n \times 1$ vector of the averages of measurements of X_q with the j^{th} observation, $\bar{x}_{q_{j.}} = \sum_{k=1}^{r_j} x_{q_{jk}} / r_j$; $\mathbf{z}_q$ as the $n \times 1$ vector of the q^{th} imprecisely observable variable, $q = s_1+1, s_1+2,...,s$; and $\bar{\mathbf{u}}_q$ as the $n \times 1$ vector of the averages of the heterogeneous random measurement errors with the j^{th} observation, $\bar{u}_{q_{j.}} = \sum_{k=1}^{r_j} u_{q_{jk}} / r_j$, $q = s_1+1, s_1+2,...,s$. The heterogeneous random measurement errors of the k^{th} measurements of X_q at the j^{th} observation, $u_{q_{jk}}$'s, $k = 1,2,...,r_j$, are independent across the measurements and distributed as $N\left(0, \sigma^2_{uq_j}\right)$, where the variance $\sigma^2_{uq_j}$ is assumed to be unknown and invariant at the measurement of the j^{th} observation.

The matrix notation in (1) can be re-written in terms of vectors as

$$\mathbf{Y} = \left[\bar{\mathbf{y}}_1 \quad \bar{\mathbf{y}}_2 \quad \dots \quad \bar{\mathbf{y}}_n\right]', \; \bar{\mathbf{y}}_j = \left[\bar{y}_{1_{j.}} \quad \bar{y}_{2_{j.}} \quad \dots \quad \bar{y}_{p_{j.}}\right]',$$

$$j = 1,2,...,n, \; \mathbf{Z} = \left[\mathbf{z}_1 \quad \mathbf{z}_2 \quad \dots \quad \mathbf{z}_n\right]',$$

$$\mathbf{z}_j = \left[z_{0_j} \quad z_{1_j} \quad \dots \quad z_{s_{1_j}} \quad z_{(s_1+1)_j} \quad \dots \quad z_{s_j}\right]',$$

$$j = 1,2,...,n, \; \mathbf{B} = \left[\boldsymbol{\beta}_0 \quad \boldsymbol{\beta}_1 \quad \dots \quad \boldsymbol{\beta}_s\right]',$$

$$\boldsymbol{\beta}_q = \left[\beta_{q1} \quad \beta_{q2} \quad \dots \quad \beta_{qp}\right]', \; q = 0,1,2,...,s,$$

$$\mathbf{E} = \left[\bar{\boldsymbol{\varepsilon}}_1 \quad \bar{\boldsymbol{\varepsilon}}_2 \quad \dots \quad \bar{\boldsymbol{\varepsilon}}_n\right]', \; \bar{\boldsymbol{\varepsilon}}_j = \left[\bar{\varepsilon}_{1_{j.}} \quad \bar{\varepsilon}_{2_{j.}} \quad \dots \quad \bar{\varepsilon}_{p_{j.}}\right]',$$

$$j = 1,2,...,n.$$

Elements of the random error vector $\bar{\boldsymbol{\varepsilon}}_j$ are independent and identically distributed as

$N_p\left(0, \boldsymbol{\Sigma}_j\right)$. The $p \times p$ variance-covariance matrix of $\bar{\boldsymbol{\varepsilon}}_j$ is assumed to be unknown and can be estimated by

$$\hat{\boldsymbol{\Sigma}}_j = \begin{bmatrix} \hat{\bar{\sigma}}_{11j} & \cdots & \hat{\bar{\sigma}}_{1pj} \\ \vdots & \ddots & \vdots \\ \hat{\bar{\sigma}}_{p1j} & \cdots & \hat{\bar{\sigma}}_{ppj} \end{bmatrix}, \; j = 1,2,...,n,$$

where $\hat{\bar{\sigma}}_{ii'j} = \hat{\sigma}_{ii'jk} / r_j$, $i, i' = 1,2,\dots,p$, $k = 1,2,\dots,r_j$.

2.2 The parameter estimation method using the corrected score approach

The first subsection of this section is the parameter estimation based on the CS approach in the general case of a linear multivariate multiple regression with HME. The parameter estimation when the covariance matrix of random errors is invariant is presented in the second subsection and the estimators are shown to be asymptotically unbiased in a specific case when there are only two independent variables, one precisely observed and another observed via a surrogate variable.

2.2.1 General case of linear multivariate multiple regression

Let $\mathbf{X}$ be a surrogate for independent variables $\mathbf{Z}$ in model (1) with the last $(s - s_1)$ imprecisely observed variables, and be expressed as

$\mathbf{X} = \left[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n \right]'$,

$\mathbf{x}_j = \left[z_{0j} \quad z_{1j} \quad \cdots \quad z_{s_{1j}} \quad \bar{x}_{(s_1+1)j.} \quad \bar{x}_{(s_2+2)j.} \quad \cdots \quad \bar{x}_{s_j.} \right]'$,

$j = 1, 2, ..., n$.

Let $L(\mathbf{B}|\mathbf{Z},\mathbf{Y})$ be the likelihood function of $\mathbf{B}$ given $\mathbf{Z}$ and $\mathbf{Y}$, $l(\mathbf{B}|\mathbf{Z},\mathbf{Y})$ be the log-likelihood function of $\mathbf{B}$ given $\mathbf{Z}$ and $\mathbf{Y}$, and $U(\mathbf{B}|\mathbf{Z},\mathbf{Y})$ be the score function of $\mathbf{B}$ given $\mathbf{Z}$ and $\mathbf{Y}$. Then the estimation of the parameters in the model in (1) and (2) using the CS approach (Nakamura, 1990; Giménez and Galae, 2013) can be described briefly as follows:

1) Construct a corrected log-likelihood function, $l_c(\mathbf{B}|\mathbf{x}_j,\bar{\mathbf{y}}_j)$, which becomes an unbiased estimating function in the absence of an imprecise measurement and satisfies the following condition:

$$E\left[l_c\left(\mathbf{B}|\mathbf{x}_j,\bar{\mathbf{y}}_j\right)\right] = l\left(\mathbf{B}|\mathbf{z}_j,\bar{\mathbf{y}}_j\right). \tag{3}$$

2) Evaluate the CS functions of $\mathbf{B}$ under regularity conditions, which yields a set of CS functions given by

$$U_{c\boldsymbol{\beta}_{qj}}\left(\mathbf{B}|\mathbf{x}_j,\bar{\mathbf{y}}_j\right) = \frac{\partial l_c\left(\mathbf{B}|\mathbf{x}_j,\bar{\mathbf{y}}_j\right)}{\partial \boldsymbol{\beta}_q},\ q = 0,1,...,s. \tag{4}$$

3) The CS estimators of $\mathbf{B}$ are determined by $E\left[U_c(\mathbf{B}|\mathbf{X},\mathbf{Y})\right] = 0$ and $E\left[U_c(\mathbf{B}|\mathbf{X},\mathbf{Y})\right] = U(\mathbf{B}|\mathbf{Z},\mathbf{Y})$, where $U_c(\mathbf{B}|\mathbf{X},\mathbf{Y})$ is an unbiased score function of $U(\mathbf{B}|\mathbf{Z},\mathbf{Y})$. By applying the general theory of M-estimation (Carroll *et al.*, 2006), Nakamura (1990) showed that

$$\sum_{j=1}^{n} U_{c\hat{\boldsymbol{\beta}}_{qj}}\left(\hat{\mathbf{B}}|\mathbf{x}_j,\bar{\mathbf{y}}_j\right) = 0,\ q = 0,1,...,s, \tag{5}$$

where $\hat{\mathbf{B}} = \left[\hat{\boldsymbol{\beta}}_0 \quad \hat{\boldsymbol{\beta}}_1 \quad ... \quad \hat{\boldsymbol{\beta}}_s\right]'$ is a matrix of the CS estimators of $\mathbf{B}$ and $\hat{\boldsymbol{\beta}}_q = \left[\hat{\beta}_{q1} \quad \hat{\beta}_{q2} \quad \cdots \quad \hat{\beta}_{qp}\right]'$, $q = 0,1,2,...,s$, with a consistent, asymptotically normal sequence of solutions.

Based on the model described in (1) and (2), the log-likelihood function is given by

$$l(\mathbf{B}|\mathbf{X},\mathbf{Y}) = l(\mathbf{B}|\mathbf{Z},\mathbf{Y}) + l(\mathbf{Z},\mathbf{X}). \tag{6}$$

Next, the score function can be written from (6) as

$$\boldsymbol{U}(\mathbf{B}|\mathbf{X},\mathbf{Y}) = \frac{\partial l(\mathbf{B}|\mathbf{X},\mathbf{Y})}{\partial \boldsymbol{\beta}_q} = \sum_{j=1}^{n} \frac{\partial l(\mathbf{B}|\mathbf{z}_j,\bar{\mathbf{y}}_j)}{\partial \boldsymbol{\beta}_q}, \text{ where}$$

$$\mathbf{z}_j = \left[z_{0j} \quad z_{1j} \quad \cdots \quad z_{s_{1j}} \quad \left(\bar{x}_{(s_1+1)j.} - \bar{u}_{(s_1+1)j.}\right) \quad \left(\bar{x}_{(s_1+2)j.} - \bar{u}_{(s_1+2)j.}\right) \quad \cdots \quad \left(\bar{x}_{sj.} - \bar{u}_{sj.}\right) \right]'. \tag{7}$$

The likelihood function $L(\mathbf{B}|\mathbf{Z},\mathbf{Y})$ of $\mathbf{B}$ given $\mathbf{Z}$ and $\mathbf{Y}$ is defined as

$$L(\mathbf{B}|\mathbf{Z},\mathbf{Y}) = \prod_{j=1}^{n} \frac{1}{(2\pi)^{p/2}} \cdot \frac{1}{\left|\boldsymbol{\Sigma}_j\right|^{1/2}} \cdot \exp\left\{ -\frac{1}{2}\left(\bar{\mathbf{y}}_j - \mathbf{B}'\mathbf{z}_j\right)' \boldsymbol{\Sigma}_j^{-1} \left(\bar{\mathbf{y}}_j - \mathbf{B}'\mathbf{z}_j\right) \right\}. \tag{8}$$

From (8), the log likelihood function of $\mathbf{B}$ given $\mathbf{z}_j$ and $\bar{\mathbf{y}}_j$ can be expressed as

$$l\left(\mathbf{B}|\mathbf{z}_j,\bar{\mathbf{y}}_j\right) = c_1 + c_2 - \frac{1}{2}\left(\bar{\mathbf{y}}_j - \mathbf{B}'\mathbf{z}_j\right)' \boldsymbol{\Sigma}_j^{-1} \left(\bar{\mathbf{y}}_j - \mathbf{B}'\mathbf{z}_j\right), \tag{9}$$

where $c_1 = -\frac{p}{2}\log(2\pi)$ and $c_2 = -\frac{1}{2}\log\left|\boldsymbol{\Sigma}_j\right|$.

Let $l\left(\mathbf{B}\middle|\mathbf{x}_j,\bar{\mathbf{y}}_j\right)$ be the log-likelihood function by substituting $\mathbf{z}_j$ in (7) into (9). Taking the expectation of $l\left(\mathbf{B}\middle|\mathbf{x}_j,\bar{\mathbf{y}}_j\right)$ and using the relationship $E\left(\bar{x}_{q_{j\cdot}}\right)=z_{q_j}$, $q=s_1+1,s_1+2,...,s$, yield

$$E\left[l\left(\mathbf{B}\middle|\mathbf{x}_j,\bar{\mathbf{y}}_j\right)\right]=c_1+c_2-\frac{1}{2}\left\{\left(\bar{\mathbf{y}}_j-\mathbf{B}'\mathbf{z}_j\right)'\boldsymbol{\Sigma}_j^{-1}\left(\bar{\mathbf{y}}_j-\mathbf{B}'\mathbf{z}_j\right)+\sum_{q=s_1+1}^{s}\boldsymbol{\beta}_q'\boldsymbol{\Sigma}_j^{-1}\boldsymbol{\beta}_q\left(\sigma_{uq_j}^2\big/r_j\right)\right\}.$$

Thus, the corrected log-likelihood function satisfying (3) can be written as

$$l_c\left(\mathbf{B}\middle|\mathbf{x}_j,\bar{\mathbf{y}}_j\right)=c_1+c_2-\frac{1}{2}\left\{\left(\bar{\mathbf{y}}_j-\mathbf{B}'\mathbf{x}_j\right)'\boldsymbol{\Sigma}_j^{-1}\left(\bar{\mathbf{y}}_j-\mathbf{B}'\mathbf{x}_j\right)-\sum_{q=s_1+1}^{s}\boldsymbol{\beta}_q'\boldsymbol{\Sigma}_j^{-1}\boldsymbol{\beta}_q\left(\sigma_{uq_j}^2\big/r_j\right)\right\}. \quad (10)$$

It can be deduced from (4) and (10) that the CS function can be expressed as

$$U_{c\boldsymbol{\beta}_{q_j}}\left(\mathbf{B}\middle|\mathbf{x}_j,\bar{\mathbf{y}}_j\right)=\begin{cases}\boldsymbol{\Sigma}_j^{-1}\left[\left(\bar{\mathbf{y}}_j-\mathbf{B}'\mathbf{x}_j\right)z_{(s_1+1)_{j\cdot}}\right], \\ q=0,1,...,s_1, \\ \boldsymbol{\Sigma}_j^{-1}\left[\left(\bar{\mathbf{y}}_j-\mathbf{B}'\mathbf{x}_j\right)\bar{x}_{q_{j\cdot}}+\boldsymbol{\beta}_q\left(\sigma_{uq_j}^2/r_j\right)\right], \\ q=s_1+1,s_1+2,...,s.\end{cases} \quad (11)$$

From (5) and (11), the $p(s+1)$ estimating equations can be written as

$$\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}\left(\hat{\mathbf{B}}'\mathbf{x}_j\right)=\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}\bar{\mathbf{y}}_j$$

$$\sum_{j=1}^{n}z_{1_j}\boldsymbol{\Sigma}_j^{-1}\left(\hat{\mathbf{B}}'\mathbf{x}_j\right)=\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}z_{1_j}\bar{\mathbf{y}}_j$$

$$\vdots$$

$$\sum_{j=1}^{n}z_{s_{1_j}}\boldsymbol{\Sigma}_j^{-1}\left(\hat{\mathbf{B}}'\mathbf{x}_j\right)=\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}z_{s_{1_j}}\bar{\mathbf{y}}_j$$

$$\sum_{j=1}^{n}\left(\bar{x}_{(s_1+1)_{j\cdot}}\boldsymbol{\Sigma}_j^{-1}\left(\hat{\mathbf{B}}'\mathbf{x}_j\right)+\left(\sigma_{u(s_1+1)_j}^2\big/r_j\right)\hat{\boldsymbol{\beta}}_{(s_1+1)}\right)=\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}\bar{x}_{(s_1+1)_{j\cdot}}\bar{\mathbf{y}}_j$$

$$\vdots$$

$$\sum_{j=1}^{n}\left(\bar{x}_{s_{j\cdot}}\boldsymbol{\Sigma}_j^{-1}\left(\hat{\mathbf{B}}'\mathbf{x}_j\right)+\left(\sigma_{us_j}^2\big/r_j\right)\hat{\boldsymbol{\beta}}_s\right)=\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}\bar{x}_{s_{j\cdot}}\bar{\mathbf{y}}_j. \quad (12)$$

Solving the $p(s+1)$ linear equations in (12) yields

$$vec\left(\hat{\mathbf{B}}_{cs}\right)=\left[\left\{\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)\right\}+\mathbf{C}\right]^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}vec\left(\mathbf{Y}'\right), \quad (13)$$

which can be expressed in the term of the OLS estimator with HME, $\hat{\mathbf{B}}_{ols/hme}$, as

$$vec\left(\hat{\mathbf{B}}_{cs}\right)=\left[\mathbf{I}_{p(s+1)}-\boldsymbol{\Psi}\right]vec\left(\hat{\mathbf{B}}_{ols/hme}\right), \quad (14)$$

where $\boldsymbol{\Psi}=\left(\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)\right)^{-1}\mathbf{C}\left(\mathbf{I}_{p(s+1)}+\left(\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)\right)^{-1}\mathbf{C}\right)^{-1}$, $\mathbf{V}^{-1}$ is the estimates of $diag\left(\boldsymbol{\Sigma}_1^{-1}\ \ \boldsymbol{\Sigma}_2^{-1}\ \ \ldots\ \ \boldsymbol{\Sigma}_n^{-1}\right)$ of size np, $\mathbf{C}$ is a block-diagonal matrix of size $p(s+1)$ where the first (s_1+1) diagonal square sub-matrices of size *p* are zero and the last $(s-s_1)$ diagonal square sub-matrices of size *p* are the estimates of $-\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}\left(\sigma_{u(s_1+1)_j}^2\big/r_j\right),...,-\sum_{j=1}^{n}\boldsymbol{\Sigma}_j^{-1}\left(\sigma_{us_j}^2\big/r_j\right)$, respectively, and

$$vec\left(\hat{\mathbf{B}}_{ols/hme}\right)=\left(\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)\right)^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}vec\left(\mathbf{Y}'\right).$$

The bias and variance of the CS estimators in (14) are respectively given by

$$Bias\left[vec\left(\hat{\mathbf{B}}_{cs}\right)\right]=$$
$$Bias\left[vec\left(\hat{\mathbf{B}}_{ols/hme}\right)\right]-E\left[(\mathbf{\Psi})vec\left(\hat{\mathbf{B}}_{ols/hme}\right)\right],$$
$$Var\left[vec\left(\hat{\mathbf{B}}_{cs}\right)\right]=Var\left[\left\{\mathbf{I}_{p(s+1)}-\mathbf{\Psi}\right\}vec\left(\hat{\mathbf{B}}_{ols/hme}\right)\right]$$

2.2.2 Invariant covariance matrix of the random error

Consider the special case where the covariance matrix of the random error is invariant, i.e. $\mathbf{\Sigma}_j = \mathbf{\Sigma}, \forall j = 1,2,...,n$. An example of the real-life application of this special case is the study of cholesterol as a function of blood pressure and weight. The measurement of blood pressure is objected to HME, depending on the time of measurement and the physical activity as well as the emotional condition of the patient but the weight can be precisely measured. The random error associated with the dependent variable, cholesterol, is assumed to be homogeneous across the observations. In this case, the term $\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}$ in (13) can be easily reduced to

$$\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_{1_1} & z_{1_2} & \cdots & z_{1_n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{s_{1_1}} & z_{s_{1_2}} & \cdots & z_{s_{1_n}} \\ \bar{x}_{(s_1+1)_{1.}} & \bar{x}_{(s_1+1)_{2.}} & \cdots & \bar{x}_{(s_1+1)_{n.}} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{x}_{s_{1.}} & \bar{x}_{s_{2.}} & \cdots & \bar{x}_{s_{n.}} \end{bmatrix}\otimes\hat{\mathbf{\Sigma}}^{-1}, \tag{15}$$

which leads to express the term $\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)+\mathbf{C}$ as

$$\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)+\mathbf{C}$$

$$=\begin{bmatrix} n & \sum_{j=1}^n z_{1_j} & \cdots & \sum_{j=1}^n z_{s_{1_j}} & \sum_{j=1}^n \bar{x}_{(s_1+1)_{j.}} & \cdots & \sum_{j=1}^n \bar{x}_{s_{j.}} \\ \sum_{j=1}^n z_{1_j} & \sum_{j=1}^n z_{1_j}^2 & \cdots & \sum_{j=1}^n z_{1_j} z_{s_{1_j}} & \sum_{j=1}^n z_{1_j}\bar{x}_{(s_1+1)_{j.}} & \cdots & \sum_{j=1}^n z_{1_j}\bar{x}_{s_{j.}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n z_{s_{1_j}} & \sum_{j=1}^n z_{s_{1_j}} z_{1_j} & \cdots & \sum_{j=1}^n z_{s_{1_j}}^2 & \sum_{j=1}^n z_{s_{1_j}}\bar{x}_{(s_1+1)_{j.}} & \cdots & \sum_{j=1}^n z_{s_{1_j}}.\bar{x}_{s_{j.}} \\ \sum_{j=1}^n \bar{x}_{(s_1+1)_{j.}} & \sum_{j=1}^n \bar{x}_{(s_1+1)_{j.}} z_{1_j} & \cdots & \sum_{j=1}^n \bar{x}_{(s_1+1)_{j.}} z_{s_{1_j}} & \sum_{j=1}^n \bar{x}_{(s_1+1)_{j.}}^2 - \sum_{j=1}^n\left(S_{u(s_1+1)_j}^2/r_j\right) & \cdots & \sum_{j=1}^n \bar{x}_{(s_1+1)_{j.}}\bar{x}_{s_{j.}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n \bar{x}_{s_{j.}} & \sum_{j=1}^n \bar{x}_{s_{j.}} z_{1_j} & \cdots & \sum_{j=1}^n \bar{x}_{s_{j.}} z_{s_{1_j}} & \sum_{j=1}^n \bar{x}_{s_{j.}}\bar{x}_{(s_1+1)_{j.}} & \cdots & \sum_{j=1}^n \bar{x}_{s_{j.}}^2 - \sum_{j=1}^n\left(S_{us_j}^2/r_j\right) \end{bmatrix}\otimes\hat{\mathbf{\Sigma}}^{-1}$$

$$=\left(\mathbf{X}'\mathbf{X}+\mathbf{C}_u\right)\otimes\hat{\mathbf{\Sigma}}^{-1}, \tag{16}$$

where $\mathbf{C}_u$ is a diagonal matrix of size $(s+1)$ where the first (s_1+1) diagonal elements are zero and the last $(s-s_1)$ elements are the estimates of $-\sum_{j=1}^n\left(\sigma_{u(s_1+1)_j}^2/r_j\right), -\sum_{j=1}^n\left(\sigma_{u(s_1+2)_j}^2/r_j\right), \ldots, -\sum_{j=1}^n\left(\sigma_{us_j}^2/r_j\right)$.

From (16), the inverse of $\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)+\mathbf{C}$ can be expressed as

$$\left(\left(\mathbf{X}\otimes\mathbf{I}_p\right)'\mathbf{V}^{-1}\left(\mathbf{X}\otimes\mathbf{I}_p\right)+\mathbf{C}\right)^{-1}=\left(\mathbf{X}'\mathbf{X}+\mathbf{C}_u\right)^{-1}\otimes\hat{\mathbf{\Sigma}}. \tag{17}$$

For simplicity of notation, let $vec(\mathbf{F}) = (\mathbf{X} \otimes \mathbf{I}_p)' \mathbf{V}^{-1} vec(\mathbf{Y}')$. Substituting $(\mathbf{X} \otimes \mathbf{I}_p)' \mathbf{V}^{-1}$ in (15) into the right hand side of the definition of $vec(\mathbf{F})$ yields

$$vec(\mathbf{F}) = vec\left[\hat{\boldsymbol{\Sigma}}^{-1}\left(\sum_{j=1}^{n} z_{0_j}\bar{\mathbf{y}}_j \quad \sum_{j=1}^{n} z_{1_j}\bar{\mathbf{y}}_j \quad \cdots \quad \sum_{j=1}^{n} z_{s_{1_j}}\bar{\mathbf{y}}_j \quad \sum_{j=1}^{n} \bar{x}_{(s_1+1)_j}\bar{\mathbf{y}}_j \quad \cdots \quad \sum_{j=1}^{n} \bar{x}_{s_j}\bar{\mathbf{y}}_j\right)\right]. \tag{18}$$

Substituting (17) and (18) into (13) gives

$$vec(\hat{\mathbf{B}}_{cs}) = vec\left(\hat{\boldsymbol{\Sigma}}\mathbf{F}(\mathbf{X}'\mathbf{X} + \mathbf{C}_u)^{-1}\right)$$

$$= vec\left(\left(\sum_{j=1}^{n} z_{0_j}\bar{\mathbf{y}}_j \quad \sum_{j=1}^{n} z_{1_j}\bar{\mathbf{y}}_j \quad \cdots \quad \sum_{j=1}^{n} z_{s_{1_j}}\bar{\mathbf{y}}_j \quad \sum_{j=1}^{n} \bar{x}_{(s_1+1)_j}\bar{\mathbf{y}}_j \quad \cdots \quad \sum_{j=1}^{n} \bar{x}_{s_j}\bar{\mathbf{y}}_j\right)(\mathbf{X}'\mathbf{X} + \mathbf{C}_u)^{-1}\right) \tag{19}$$

where the i^{th} CS estimator, $\hat{\boldsymbol{\beta}}_{i\_cs}$, $i = 1, 2, \ldots, p$ can be expressed as

$$\hat{\boldsymbol{\beta}}_{i\_cs} = (\mathbf{X}'\mathbf{X} + \mathbf{C}_u)^{-1}\mathbf{X}'\bar{\mathbf{y}}_i, \tag{20}$$

where $\bar{\mathbf{y}}_i = \begin{bmatrix}\bar{y}_{i_{1.}} & \bar{y}_{i_{2.}} & \cdots & \bar{y}_{i_{n.}}\end{bmatrix}'$, $i = 1, 2, ..., p$. The $\hat{\boldsymbol{\beta}}_{i\_cs}$ can be reduced to the same results given by Giménez and Patat (2005) when the MEs are homogeneous. In this study, the heterogeneous variance of the MEs is unknown and it is being estimated by the pooled sample variance. In the case of grouped heteroscedasticity, the observations are grouped into several subsets such that the variance of the MEs is homogeneous within a group but heterogeneous across the groups (Judge *et al.*, 1985). Let the number of groups be g, the size of the h^{th} group be n_h, $h = 1, 2, ..., g$, σ^2_{uqh} be the homogeneous variance of the MEs of X_q in the h^{th} group and r_h be the number of repeated measurements of each observation in the h^{th} group. Then the sample variance of X_q in the h^{th} group is given by

$$S^2_{uqh} = \frac{\sum_{j_h=1}^{n_h}\sum_{k=1}^{r_h}\left(x_{q_{j_h k}} - \bar{x}_{q_{j_h .}}\right)^2}{n_h r_h}.$$

Therefore, the q^{th} diagonal element of $\mathbf{C}_u$, $-\sum_{j=1}^{n} S^2_{uq_j} / r_j$, can be estimated by the pooled variance as $-\sum_{h=1}^{g} n_h S^2_{uqh} / r_h$, $q = s_1 + 1, s_1 + 2, \ldots, s$.

The estimator directly obtained by the OLS method without score correcting is the case in (20) where $\mathbf{C}_u$ is a zero matrix. Thus, from (20), the $\hat{\boldsymbol{\beta}}_{i\_ols/hme}$ estimator can be expressed as

$$\hat{\boldsymbol{\beta}}_{i\_ols/hme} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\bar{\mathbf{y}}_i. \tag{21}$$

Consider the specific case where $s_1 = 1$, $s = 2$, i.e. the independent variable Z_1 is precisely observed and the independent variable Z_2 is measured by the surrogate X_2 with HME. Then, the CS estimators in (19) yields

$$vec(\hat{\mathbf{B}}_{cs}) = vec\left(\left(\sum_{j=1}^{n} z_{0_j}\bar{\mathbf{y}}_j \quad \sum_{j=1}^{n} z_{1_j}\bar{\mathbf{y}}_j \quad \sum_{j=1}^{n} \bar{x}_{2_j.}\bar{\mathbf{y}}_j\right)(\mathbf{X}'\mathbf{X} + \mathbf{C}_u)^{-1}\right). \tag{22}$$

The i^{th} vector in $vec(\hat{\mathbf{B}}_{cs})$ in (22) can be written as

$$\hat{\boldsymbol{\beta}}_{i\_cs} = (\mathbf{X}'\mathbf{X} + \mathbf{C}_u)^{-1}\mathbf{X}'\bar{\mathbf{y}}_i, \tag{23}$$

where $\hat{\boldsymbol{\beta}}_{i\_cs} = \begin{bmatrix}\hat{\beta}_{0i} & \hat{\beta}_{1i} & \hat{\beta}_{2i}\end{bmatrix}'$, $\bar{\mathbf{y}}_i = \begin{bmatrix}\bar{y}_{i_{1.}} & \bar{y}_{i_{2.}} & \cdots & \bar{y}_{i_{n.}}\end{bmatrix}'$, $i = 1, 2, ..., p$,

$$\mathbf{X} = \begin{bmatrix} 1 & z_{1_1} & \bar{x}_{2_{1.}} \\ 1 & z_{1_2} & \bar{x}_{2_{2.}} \\ \vdots & \vdots & \vdots \\ 1 & z_{1_n} & \bar{x}_{2_{n.}} \end{bmatrix}, \quad \mathbf{C}_u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -A \end{bmatrix} \text{ and } A = \sum_{h=1}^{g} \frac{n_h S^2_{u2h}}{r_h}.$$

Lemma 1. In a grouped heteroscedasticity, the n observations can be grouped into h groups such that the variance of the measurement errors, σ^2_{u2h}, is homogeneous within the h^{th} group but heterogeneous across the groups. Let r_j be the number of repeated measurements of the j^{th} observation and u_{2jk} be the random measurement error of the j^{th} observation of x_{2j} in the k^{th} repeated measurement independently distributed as $N(0,\sigma^2_{u2_j})$. Then, as $n \to \infty$,

$$\sum_{j=1}^{n} \bar{u}^2_{2j.} = n\left(S^2_{\bar{u}_2} + \bar{u}^2_2\right) \to \sum_{h=1}^{g} \frac{n_h \sigma^2_{u2h}}{r_h}.$$

Theorem 1. In the linear multivariate measurement error regression model described in (1) and (2) where $s_1 = 1$ and $s = 2$, $\sigma^2_{\bar{u}_2} \ll \sigma^2_{z_2}$ and $\bar{u}^2_2 \ll \sigma^2_{z_2}$, $\hat{\beta}_{1i\_cs}$ is an unbiased estimator but $\hat{\beta}_{0i\_cs}$ and $\hat{\beta}_{2i\_cs}$ are asymptotically unbiased estimators, $i = 1, 2, \ldots, p$.

In summary, the estimators $\hat{\beta}_{1i\_ols/hme}$ and $\hat{\beta}_{1i\_cs}$ are both unbiased. The biases of $\hat{\beta}_{0i\_ols/hme}$ and $\hat{\beta}_{2i\_ols/hme}$ asymptotically approach to $\frac{\beta_{2i}\bar{z}_2\sigma^2_{\bar{u}_2}}{S^2_{z_2}}$, and $-\frac{\beta_{2i}\sigma^2_{\bar{u}_2}}{S^2_{z_2}}$, respectively. In the case that $\beta_{2i} > 0$, $\hat{\beta}_{0i\_ols/hme}$ may be either an overestimated or underestimated parameter depending on the signs of β_{0i} and $\bar{z}_2$ whereas $\hat{\beta}_{2i\_ols/hme}$ is definitely an underestimated parameter. On the other hand, $\hat{\beta}_{0i\_cs}$ and $\hat{\beta}_{2i\_cs}$ are asymptotically unbiased estimators.

The proofs of Lemma 1 and Theorem 1 are included in Appendix A.

3. Simulation Study

The objective of the simulation study is to empirically analyze the parameter estimations by the OLS and CS methods when varying the sample size n and the number of repeated measurements at the j^{th} observation, r_j. The proposed CS estimator is compared to the OLS estimator by considering bias and mean square error (MSE). Data sets are generated from the model defined in (1) and (2) with two dependent variables $(p = 2)$ and two independent variables $(s = 2)$. One of the independent variables, Z_1, is precisely observable and is generated with the uniform distribution $U[-1,1]$ to allow a small variation in Z_1 whereas the other, Z_2, cannot be precisely observed and is generated with the standard normal distribution $N(0,1)$ to take into account the random error in measurement. The parameters in the model are set as follows: $\beta_{0i} = 0$, $\beta_{1i} = \beta_{2i} = 1$, $i = 1$ and 2 and the variance-covariance matrix is set as $\mathbf{\Sigma}_j = \begin{pmatrix} 0.8 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$, $j = 1, 2, \ldots, n$. The surrogate variable X_2 instead of Z_2 is observed with the same r repeated measurements at each observation. The observations for each sample size are grouped into five sub-samples such that the variance of the random ME is homogeneous within a group but heterogeneous between groups. The HME variance in the simulation is set in two forms: the step-up function form $(F1)$ and the step-down function form $(F2)$, as specified in Table 1, and are referred to as HME forms. Each HME form is grouped into five sub-samples of equal size $(h = 1, 2, 3, 4, 5)$. The random measurement error u_2 in the h^{th} group is distributed as $N\left(0, \sigma^2_{u2h}\right)$. In the simulation, three sample sizes, n, are specified: 50, 100 and 500, and the number of repeated measurements, r: 5, 10, 20 and 40. One hundred replications are simulated for a particular combination of n and r.

Table 1. 4HME Variances in HME Forms $F1$ and $F2$

HME Form	HME Variance				
	σ^2_{u21}	σ^2_{u22}	σ^2_{u23}	σ^2_{u24}	σ^2_{u25}
$F1$	0.1	0.2	0.4	0.6	0.8
$F2$	0.8	0.6	0.4	0.2	0.1

3.1 Simulation Results

Before the analysis of the simulation data, the means of the estimated sample variances, S^2_{u2h}, of the MEs in each group in HME form $F1$ and $F2$ are tested with the given values in Table 1. The results of the estimation of HME variance based on grouped heteroscedasticity reveal that the null hypothesis of the population parameter σ^2_{u2h} value in Table 1 is not rejected at the .05 significance level. The number of replications, r, does not affect the magnitude of the bias of S^2_{u2h}. The value of S^2_{u2h} approaches to the population parameter σ^2_{u2h} when the sample size n increases, and the standard error decreases when either r or n increases. The results of the estimators under HME forms $F1$ and $F2$ reach the same conclusions. Thus, the tables of the simulation results are shown only in the case of HME form $F1$.

Tables 2 and 3 summarize the statistics and MSEs of the CS and OLS estimators under HME form $F1$. Table 2 shows the sample mean, standard error (SE), *p*-value, bias, and MSE of the parameter estimator $\hat{\beta}_{1i}$, $i = 1$ and 2, of the precisely observed variable Z_1. For any given values of r or n, the sample means of $\hat{\beta}_{1i}$, $i = 1$ and 2, estimated by the OLS and CS methods are not significantly different from the true value of parameter β_{1i}, $i = 1$ and 2, with $p > .05$. The bias and MSE of the parameter estimator $\hat{\beta}_{1i}$, $i = 1$ and 2, of the precisely observed variable, Z_1, are practically indifferent in the OLS and CS methods as shown by the ratios of the SE, absolute bias and MSE in the OLS method over the proposed CS method in the last three columns in Table 2.

Table 2. Statistics and MSEs of $\hat{\beta}_{11}$, $\hat{\beta}_{12}$ under HME Form $F1$.

n	r	Parameter	Sample Mean		SE		Bias		MSE		Ratio of $abs(OLS)/abs(CS)$		
			OLS	CS	OLS	CS	OLS	CS	OLS	CS	SE	Bias	MSE
50	5	β_{21}	0.9984	0.9986	0.0102	0.0103	-0.0016	-0.0014	0.0103	0.0104	0.9971	1.1429	0.9933
		β_{22}	0.9945	0.9946	0.0115	0.0117	-0.0055	-0.0054	0.0132	0.0135	0.9889	1.0185	0.9785
	10	β_{21}	1.0067	1.0066	0.0070	0.0070	0.0067	0.0066	0.0049	0.0049	0.9971	1.0152	0.9959
		β_{22}	0.9997	0.9996	0.0077	0.0077	-0.0003	-0.0004	0.0059	0.0059	0.9935	0.7500	0.9865
	20	β_{21}	1.0023	1.0023	0.0052	0.0052	0.0023	0.0023	0.0026	0.0027	0.9961	1.0000	0.9925
		β_{22}	1.0007	1.0007	0.0054	0.0054	0.0007	0.0007	0.0029	0.0029	0.9963	1.0000	0.9932
	40	β_{21}	0.9979	0.9978	0.0032	0.0032	-0.0021	-0.0022	0.0010	0.0010	1.0000	0.9545	1.0000
		β_{22}	0.9970	0.9970	0.0039	0.0039	-0.0030	-0.0030	0.0016	0.0016	1.0000	1.0000	1.0000
100	5	β_{21}	0.9959	0.9962	0.0067	0.0067	-0.0041	-0.0038	0.0044	0.0044	1.0030	1.0789	1.0045
		β_{22}	0.9988	0.9990	0.0077	0.0078	-0.0012	-0.0010	0.0058	0.0059	0.9846	1.2000	0.9949
	10	β_{21}	1.0014	1.0013	0.0052	0.0052	0.0014	0.0013	0.0027	0.0027	0.9981	1.0769	0.9963
		β_{22}	1.0024	1.0022	0.0059	0.0059	0.0024	0.0022	0.0034	0.0035	0.9966	1.0909	0.9942
	20	β_{21}	0.9984	0.9983	0.0037	0.0037	-0.0016	-0.0017	0.0014	0.0014	0.9973	0.9412	0.9928
		β_{22}	0.9972	0.9972	0.0042	0.0042	-0.0028	-0.0028	0.0017	0.0017	0.9976	1.0000	0.9943
	40	β_{21}	0.9963	0.9962	0.0027	0.0027	-0.0037	-0.0038	0.0007	0.0007	1.0000	0.9737	1.0000
		β_{22}	1.0006	1.0006	0.0029	0.0029	0.0006	0.0006	0.0008	0.0008	0.9966	1.0167	0.9881
500	5	β_{21}	1.0005	1.0006	0.0030	0.0030	0.0005	0.0006	0.0009	0.0009	0.9933	0.8333	0.9888
		β_{22}	1.0013	1.0015	0.0034	0.0034	0.0013	0.0015	0.0001	0.0012	0.9854	0.8667	0.0948
	10	β_{21}	1.0020	1.0020	0.0024	0.0024	0.0020	0.0020	0.0006	0.0006	0.9959	1.0000	0.9828
		β_{22}	1.0012	1.0012	0.0028	0.0028	0.0012	0.0012	0.0008	0.0008	0.9964	1.0000	1.0000
	20	β_{21}	1.0006	1.0006	0.0016	0.0016	0.0006	0.0006	0.0003	0.0003	1.0000	1.0000	1.0000
		β_{22}	0.9986	0.9986	0.0018	0.0018	-0.0014	-0.0014	0.0003	0.0003	1.0000	1.0000	1.0000
	40	β_{21}	1.0005	1.0006	0.0012	0.0012	0.0005	0.0006	0.0002	0.0002	1.0000	0.8333	1.0000
		β_{22}	1.0000	1.0000	0.0012	0.0012	0.0000	0.0000	0.0002	0.0002	1.0000	1.0000	1.0000

Table 3. Statistics and MSEs of $\hat{\beta}_{21}$, $\hat{\beta}_{22}$ under HME Form $F1$.

n	r	Parameter	Sample Mean		SE		Bias		MSE		Ratio of $abs(OLS)/abs(CS)$		
			OLS	CS	OLS	CS	OLS	CS	OLS	CS	SE	Bias	MSE
50	5	β_{21}	0.9782	1.0016	0.00261	0.00265	-0.0218	0.0016	0.001148	0.00077	0.9849	13.6250	1.4909
		β_{22}	0.98	1.0034	0.00301	0.00307	-0.02	0.0034	0.001298	0.000945	0.9805	5.8824	1.3735
	10	β_{21}	0.9908	1.0019	0.00196	0.00195	-0.0092	0.0019	0.000464	0.00038	1.0051	4.8421	1.2211
		β_{22}	0.9904	1.0015	0.00207	0.00207	-0.0096	0.0015	0.000514	0.000426	1.0000	6.4000	1.2066
	20	β_{21}	0.9962	1.0018	0.00143	0.00143	-0.0038	0.0018	0.000216	0.000206	1.0000	2.1111	1.0485
		β_{22}	0.9956	1.0012	0.00166	0.00162	-0.0044	0.0012	0.000291	0.000275	1.0247	3.6667	1.0582
	40	β_{21}	0.9949	0.9977	0.00112	0.00112	-0.0051	-0.0023	0.000149	0.00013	1.0000	2.2174	1.1462
		β_{22}	0.9955	0.9983	0.00125	0.00124	-0.0045	-0.0017	0.000174	0.000155	1.0081	2.6471	1.1226
100	5	β_{21}	0.9781	0.9993	0.00223	0.00227	-0.0219	-0.0007	0.000974	0.000509	0.9824	31.2857	1.9136
		β_{22}	0.9762	0.9975	0.00216	0.00218	-0.0238	-0.0025	0.001027	0.000478	0.9908	9.5200	2.1485
	10	β_{21}	0.9882	0.9988	0.00143	0.00148	-0.0118	-0.0012	0.000341	0.000217	0.9662	9.8333	1.5714
		β_{22}	0.9882	0.9988	0.00145	0.00147	-0.0118	-0.0012	0.000347	0.000215	0.9864	9.8333	1.6140
	20	β_{21}	0.9964	1.0017	0.00107	0.00109	-0.0036	0.0017	0.000126	0.00012	0.9817	2.1176	1.0500
		β_{22}	0.996	1.0013	0.00125	0.00127	-0.004	0.0013	0.000171	0.000162	0.9843	3.0769	1.0556
	40	β_{21}	0.9979	1.0006	0.00069	0.00069	-0.0021	0.0006	0.000052	0.000047	1.0000	3.5000	1.1064
		β_{22}	0.9977	1.0004	0.00078	0.00078	-0.0023	0.0004	0.000066	0.000061	1.0000	5.7500	1.0820
500	5	β_{21}	0.9797	1.0003	0.00099	0.00102	-0.0203	0.0003	0.00051	0.000103	0.9706	67.6667	4.9515
		β_{22}	0.9797	1.0003	0.00115	0.00116	-0.0203	0.0003	0.000543	0.000134	0.9914	67.6667	4.0522
	10	β_{21}	0.9892	0.9996	0.00063	0.00064	-0.0108	-0.0006	0.000156	0.00004	0.9844	18.0000	3.9000
		β_{22}	0.9893	0.9997	0.00076	0.00077	-0.0107	-0.0007	0.000171	0.000058	0.9870	15.2857	2.9483
	20	β_{21}	0.995	1.0003	0.00047	0.00048	-0.005	0.0003	0.000046	0.000023	0.9792	16.6667	2.0000
		β_{22}	0.9948	1.0001	0.00053	0.00054	-0.0052	0.0001	0.000055	0.000029	0.9815	52.0000	1.8966
	40	β_{21}	0.9977	1.0003	0.00035	0.00036	-0.0023	0.0003	0.000018	0.000013	0.9722	7.6667	1.3846
		β_{22}	0.9972	0.9998	0.0004	0.0004	-0.0028	-0.0002	0.000024	0.000016	1.0000	14.0000	1.5000

Table 3 shows the statistics and MSEs of the parameter estimator $\hat{\beta}_{2i}$, $i = 1, 2$ of the variable Z_2 under HME. The t-test for the sample means of $\hat{\beta}_{2i}$, $i = 1$ and 2, estimated by the OLS method shows that the estimate is significantly different from the true value β_{2i}, $i = 1$ and 2, with $p < .05$, and the bias is negative. Thus, it can be concluded that the sample means of β_{2i}, $i = 1$, and 2, estimated by the OLS method are underestimated as shown in Section 2. Meanwhile, the t-test for the sample means of β_{2i}, $i = 1$, and 2, estimated by the CS method shows that the estimate is not significantly different from the true value β_{2i}, $i = 1$, and 2, with $p < .05$. The simulation confirms the analytical results of the estimator bias in Section 2. In Table 3 it can be seen that the SEs of the sample mean estimated by the OLS method and the proposed CS method are slightly different and converge to the same value when r increases. The ratios of the SE in the OLS method over the proposed CS method are between 0.9662 and 1.0247. The absolute bias and MSE in the OLS method are greater than the corresponding ones in the proposed CS method as shown by the ratios of the absolute bias and MSE in the OLS method over the proposed CS method in the last two columns in Table 3. Additionally, the bias of the CS estimator approaches to zero as n increases whereas the absolute bias of the OLS estimator decreases but does not approach to zero.

4. Conclusions

This study extends the estimation theory based on the CS to cover a linear multivariate multiple regression model consisting of s_1 precisely observed independent variables and $(s-s_1)$ independent variables with HMEs. The HME variance is assumed to be unknown and is estimated based on grouped heteroscedasticity. In each group, the variance of the MEs of the surrogate variable is estimated by the pooled variance of the variable with HMEs observed in the repeated measurements.

In the case of independently and identically distributed random errors and homogeneous measurement error, the analytical results agree with the findings of Giménez and Patat (2005). In the specific case where the multivariate regression model consists of p dependent variables, one precisely observed independent variable and one independent variable with HME, it is shown that the estimates of $\beta_{0i\_ols/hme}$ and $\beta_{2i\_ols/hme}$ are biased but the estimates of β_{0i_cs} and β_{2i_cs} are asymptotically unbiased, and that the estimates of $\beta_{1i\_ols/hme}$ and β_{1i_cs} are both unbiased for $i=1,2,\ldots,p$.

A simulation study is carried out on the model specified above. The simulation results show that the proposed CS method outperforms the OLS method in terms of MSEs of the parameter estimates and the OLS estimation of the parameters of the precisely observed variable is unaffected by HME, but the parameter estimators of the variable measured with HME are underestimated. The bias of the CS estimator approaches to zero when the sample size increases.

The current study could easily be extended to allow for non-equal numbers of repeated measurements of the surrogate variables by changing r_j, the number of repeated measurements at the j^{th} observation, to r_{q_j}, the number of repeated measurements of the q^{th} surrogate variable at the j^{th} observation. In future work, some other approaches to solving the problem of the HME variance estimation should be investigated intensively to support other types of HME.

Acknowledgements

The authors are very grateful to the National Institute of Development Administration (NIDA) and the National Science and Technology Development Agency for funding this work and would sincerely like to thank the unanimous reviewers for the constructive comments on the earlier version of this article.

References

Carroll, R. J., Ruppert, D., Stefanski, L. A., & Crainiceanu, C. M. (2006). *Measurement error in nonlinear models, a modern perspective* (2nd ed.). New York, NY: Chapman and Hall.

Buzas, J. S., Stefanski L. A., & Tosteson T. D. (2005). Measurement error. In W. Ahrens & I. Pigeot (Eds.), *Handbook of Epidemology* (pp. 729-765). Heidelberg, Germany: Springer.

Chen, J., Hanfelt, J. J., & Huang, Y. (2015). A simple corrected score for logistic regression with errors-in-covariates. *Communications in Statistics - Theory Methods*, *44*(10), 2024-2036.

Cheng, C. L., & Riu, J. (2006). On estimating linear relationships when both variables are subject to heteroscedastic measurement errors. *Technometrics*, *48*(4), 511-519.

de Castro, M., Bolfarine, H., & Castilho, M. V. (2006). Consistent estimation and testing in comparing analytical bias models. *Environmetrics, 17*(2), 167-182.

de Castro, M., Bolfarine, H., & Galea, M. (2008). Hypothesis testing in an errors-in-variables model with heteroscedastic measurement errors. *Statistics in Medicine, 27*(25), 5217-5234.

Giménez, P., & Bolfarine, H. (1997). Corrected score functions in classical error-in-variables and incidental parameters models. *The Australian Journal of Statistics. 39*(3), 325-344.

Giménez, P., & Bolfarine, H. (2000). Comparing consistent estimators in comparative calibration models. *Journal of Statistical Planning and Inference, 86*(1),143-155.

Giménez, P., & Galae, M. (2013). Influence measures on corrected score estimators in functional heteroscedastic measurement error models. *Journal of Multivariate Analysis, 114*(1), 1-15.

Giménez, P., & Patat, M. L. (2005). Estimation in comparative calibration models with replicate measurement. *Statistics and Probability Letters, 71*(2), 155-164.

Giménez, P., & Patat, M. L. (2014). Local influence for functional comparative calibration models with replicated data. *Statistical Papers, 55*(2), 431-454.

Gujarati, D. N. (2006). *Essentials of econometrics* (3rd Ed.). New York, NY: McGraw-Hill.

Huang, Y., & Wang, C. Y. (2001). Consistent functional methods for logistic regression with errors in covariates. *Journal of the American Statistical Association, 96,* 1469-1482.

Judge, G. G., Griffiths, W. E., Hill, R. C., Lütkepohl, H., & Lee, T. C. (1985). *The theory and practice of econometrics* (2nd Ed.). New York, NY: Wiley.

Kulathinal, S. B., Kuulasmaa, K., & Gasbarra, D. (2002). Estimation of an errors-in-variable regression model when the variances of the measurement errors vary between the observations. *Statistics in Medicine, 21*(8), 1089-1101.

Nakamura, T. (1990). Corrected score function for errors-in-variables models: methodology and application to generalized linear models. *Biometrika, 77*(1), 127-137.

Patriota, A. G., Bolfarine, H., & de Castro, M. (2009). A heteroscedastic structural errors-in-variables model with equation error. *Statistical Methodology, 6*(4), 408-423.

Veenendaal, E. M., Mantlana, K. B., Pammenter, N. W., Weber, P., Mapila, P. H., & Lloyd, J. (2008). Growth form and seasonal variation in leaf gas exchange of Colophospermum mopane savanna trees in northwest Botswana. *Tree Physiology, 28*(3), 417-424.

Appendix A. Proofs of Lemma 1 and Theorem 1.

Proof of Lemma 1.

The sample variance of the average of the random measurement errors can be written as

$$S_{\bar{u}_2}^2 = \frac{1}{n}\sum_{j=1}^{n}\left(\bar{u}_{2j.} - \bar{u}_2\right)^2, \tag{24}$$

which can be estimated by the pooled variance as

$$S_{\bar{u}_2}^2 = \frac{1}{n}\sum_{h=1}^{g}\frac{n_h S_{u2h}^2}{r_h}, \tag{25}$$

where S_{u2h}^2 is the sample variance of the measurement error in the h^{th} group of n_h observations, r_h is the number of repeated measurements of the observation in the h^{th} group.

From (24), the sum of squares of the average of random measurement errors of the j^{th} observation can be expressed as

$$\sum_{j=1}^{n}\bar{u}_{2j.}^2 = n\left(S_{\bar{u}_2}^2 + \bar{u}_2^2\right). \tag{26}$$

Substituting (25) into (26) yields

$$\sum_{j=1}^{n}\bar{u}_{2j.}^2 = \sum_{h=1}^{g}\frac{n_h S_{u2h}^2}{r_h} + n\bar{u}_2^2. \tag{27}$$

As $n \to \infty$, (27) becomes

$$\sum_{j=1}^{n}\bar{u}_{2j.}^2 \to \sum_{h=1}^{g}\frac{n_h \sigma_{u2h}^2}{r_h}.$$

This completes the proof of Lemma 1.

Proof of Theorem 1.

The bias of $\hat{\boldsymbol{\beta}}_{i\_cs}$ can be written from (23) as

$$\begin{aligned} Bias\ of\ \hat{\boldsymbol{\beta}}_{i\_cs} &= E\left[\left(\mathbf{X}'\mathbf{X} + \mathbf{C}_u\right)^{-1}\mathbf{X}'\bar{\mathbf{y}}_{i.}\right] - \boldsymbol{\beta}_i \\ &= E\left[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\bar{\mathbf{y}}_{i.} - \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}_u\left[\mathbf{I} + \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}_u\right]^{-1}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\bar{\mathbf{y}}_{i.}\right] - \boldsymbol{\beta}_i. \end{aligned} \tag{28}$$

The bias of the $\hat{\boldsymbol{\beta}}_{i\_ols/hme}$ estimator from (21) can be expressed as

$$\begin{aligned} Bias\ of\ \hat{\boldsymbol{\beta}}_{i\_ols/hme} &= E\left[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\bar{\mathbf{y}}_{i}\right] - \boldsymbol{\beta}_i \\ &= E\left[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\left(\mathbf{X}\boldsymbol{\beta}_i + \mathbf{v}_i\right)\right] - \boldsymbol{\beta}_i \\ &= E\left[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{v}_i\right], \end{aligned} \tag{29}$$

where $\mathbf{v}_i = \bar{\boldsymbol{\varepsilon}}_i - \beta_{2i}\bar{\mathbf{u}}_2$. Substituting (29) into (28) yields

$$Bias\ of\ \hat{\boldsymbol{\beta}}_{i\_cs} = Bias\ of\ \hat{\boldsymbol{\beta}}_{i\_ols/hme} - E\left[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}_u\left[\mathbf{I}+\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}_u\right]^{-1}\hat{\boldsymbol{\beta}}_{i\_ols/hme}\right]. \quad (30)$$

From the definition of $\mathbf{X}$ in (23) and by using the independent property of $\mathbf{z}_1$ and $\mathbf{x}$, the inverse of $\mathbf{X}'\mathbf{X}$ can be expressed in terms of the statistics of the observations as

$$\left(\mathbf{X}'\mathbf{X}\right)^{-1} = \frac{1}{n}\begin{bmatrix} \dfrac{S_{z_1}^2+\bar{z}_1^2}{S_{z_1}^2}+\dfrac{\left(\bar{z}_2+\bar{u}_2\right)^2}{S_{z_2}^2+S_{\bar{u}_2}^2} & -\dfrac{\bar{z}_1}{S_{z_1}^2} & -\dfrac{\left(\bar{z}_2+\bar{u}_2\right)}{S_{z_2}^2+S_{\bar{u}_2}^2} \\ -\dfrac{\bar{z}_1}{S_{z_1}^2} & \dfrac{1}{S_{z_1}^2} & 0 \\ -\dfrac{\left(\bar{z}_2+\bar{u}_2\right)}{S_{z_2}^2+S_{\bar{u}_2}^2} & 0 & \dfrac{1}{S_{z_2}^2+S_{\bar{u}_2}^2} \end{bmatrix},$$

which is denoted by

$$\left(\mathbf{X}'\mathbf{X}\right)^{-1} = \begin{bmatrix} q^{11} & q^{12} & q^{13} \\ q^{21} & q^{22} & q^{23} \\ q^{31} & q^{32} & q^{33} \end{bmatrix}. \quad (31)$$

Again, from the definition of $\mathbf{X}$ in (23) and the definition of $\mathbf{v}_i$ in (29), the term $\mathbf{X}'\mathbf{v}_i$ in (29) can be expressed as

$$\mathbf{X}'\mathbf{v}_i = \begin{bmatrix} \sum_{j=1}^{n}\left(\bar{\varepsilon}_{ij.}-\beta_{2i}\bar{u}_{2j.}\right) \\ \sum_{j=1}^{n} z_{1j.}\left(\bar{\varepsilon}_{ij.}-\beta_{2i}\bar{u}_{2j.}\right) \\ \sum_{j=1}^{n} \bar{x}_{2j.}\left(\bar{\varepsilon}_{ij.}-\beta_{2i}\bar{u}_{2j.}\right) \end{bmatrix},$$

which is denoted by

$$\mathbf{X}'\mathbf{v}_i = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}. \quad (32)$$

By substituting (31) and (32) into (29), the bias of the $\hat{\boldsymbol{\beta}}_{i\_ols/hme}$ becomes

$$\begin{bmatrix} Bias\ of\ \hat{\beta}_{0i\_ols/hme} \\ Bias\ of\ \hat{\beta}_{1i\_ols/hme} \\ Bias\ of\ \hat{\beta}_{2i\_ols/hme} \end{bmatrix} = E\begin{bmatrix} q^{11}d_1+q^{12}d_2+q^{13}d_3 \\ q^{21}d_1+q^{22}d_2+q^{23}d_3 \\ q^{31}d_1+q^{32}d_2+q^{33}d_3 \end{bmatrix}. \quad (33)$$

From (31), it can be easily seen that the term $\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}_u = \begin{bmatrix} 0 & 0 & -Aq^{13} \\ 0 & 0 & -Aq^{23} \\ 0 & 0 & -Aq^{33} \end{bmatrix}$ which leads to the expression of the last term in (30) as

$$E\left[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}_u\left(\mathbf{I}+\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}_u\right)^{-1}\hat{\boldsymbol{\beta}}_{i\_ols/hme}\right]=E\left(\frac{-1}{1-Aq^{33}}\right)\begin{bmatrix}Aq^{13}\hat{\beta}_{2i\_ols/hme}\\Aq^{23}\hat{\beta}_{2i\_ols/hme}\\Aq^{33}\hat{\beta}_{2i\_ols/hme}\end{bmatrix}. \tag{34}$$

Substituting the bias of $\hat{\boldsymbol{\beta}}_{i\_ols/hme}$ in (33) and the RHS in (34) into (30) yields

$$\begin{bmatrix}Bias\ of\ \hat{\beta}_{0i\_cs}\\Bias\ of\ \hat{\beta}_{1i\_cs}\\Bias\ of\ \hat{\beta}_{2i\_cs}\end{bmatrix}=E\begin{bmatrix}q^{11}d_1+q^{12}d_2+q^{13}d_3\\q^{21}d_1+q^{22}d_2+q^{23}d_3\\q^{31}d_1+q^{32}d_2+q^{33}d_3\end{bmatrix}+E\left[\left(\frac{1}{1-Aq^{33}}\right)\begin{bmatrix}Aq^{13}\hat{\beta}_{2i\_ols/hme}\\Aq^{23}\hat{\beta}_{2i\_ols/hme}\\Aq^{33}\hat{\beta}_{2i\_ols/hme}\end{bmatrix}\right]. \tag{35}$$

From the definitions of q^{ij} and d in (31) and (32), respectively, the bias of $\hat{\boldsymbol{\beta}}_{i\_ols/hme}$ in (33) can be expressed in terms of statistical properties of the variables as

$$Bias\ of\ \hat{\beta}_{0i\_ols/hme}=-\beta_{2i}E\left[\frac{\left(\bar{z}_2+\bar{u}_2\right)^2\bar{u}_2}{S_{z_2}^2\left(1+\frac{S_{\bar{u}_2}^2}{S_{z_2}^2}\right)}\right]+\beta_{2i}E\left[\frac{\bar{z}_2+\bar{u}_2}{S_{z_2}^2\left(1+\frac{S_{\bar{u}_2}^2}{S_{z_2}^2}\right)}\sum_{j=1}^{n}\frac{\bar{u}_{2j\cdot}^2}{n}\right]. \tag{36}$$

By Lemma 1, the bias of $\hat{\beta}_{0i\_ols/hme}$ in (36) can be written after the first order approximation under the assumption $S_{\bar{u}_2}^2 \ll S_{z_2}^2$ as

$$Bias\ of\ \hat{\beta}_{0i\_ols/hme}\simeq\frac{\beta_{2i}}{S_{z_2}^2}\left[\bar{z}_2\sigma_{\bar{u}_2}^2\left(1-\frac{2}{n}\right)+\frac{3\bar{z}_2}{nS_{z_2}^2}\left(\frac{2}{n}-1\right)+\frac{2\bar{z}_2}{n^2S_{z_2}^2}E\left(\sum_{j=1}^{n}\sum_{l=j+1}^{n}\bar{u}_{2j\cdot}^2\bar{u}_{2l\cdot}^2\right)\left(\frac{2}{n}-1\right)\right]. \tag{37}$$

As $n\to\infty$, the bias of $\hat{\beta}_{0i\_ols/hme}$ in (37) approaches

$$Bias\ of\ \hat{\beta}_{0i\_ols/hme}\to\frac{\beta_{2i}\bar{z}_2\sigma_{\bar{u}_2}^2}{S_{z_2}^2}. \tag{38}$$

Now consider the bias of $\hat{\beta}_{0i\_cs}$, which is given in (35) , as

$$Bias\,of\ \hat{\beta}_{0i\_cs}=Bias\,of\ \hat{\beta}_{0i\_ols/hme}+E\left(\frac{Aq^{13}\hat{\beta}_{2i\_ols/hme}}{1-Aq^{33}}\right). \tag{39}$$

With the result in (33), the last term in the RHS of (39) can be written as

$$E\left(\frac{Aq^{13}\hat{\beta}_{2i\_ols/hme}}{1-Aq^{33}}\right)=E\left[\frac{Aq^{13}\left(\beta_{2i}+q^{31}d_1+q^{32}d_2+q^{33}d_3\right)}{1-Aq^{33}}\right]. \tag{40}$$

From the definitions of q^{ij} and d_i in (31) and (32), respectively, by Lemma 1, (40) can be expressed in terms of statistical properties of the variables as

$$E\left(\frac{Aq^{13}\hat{\beta}_{2i\_ols/hme}}{1-Aq^{33}}\right) \simeq \frac{1}{S_{z_2}^2}E\left[\begin{array}{l} -\bar{z}_2 S_{\bar{u}_2}^2\beta_{2i} - \frac{(\bar{z}_2+\bar{u}_2)^2 S_{\bar{u}_2}^2\beta_{2i}\bar{u}_2}{S_{z_2}^2} + \frac{(\bar{z}_2+\bar{u}_2)^2\left(S_{\bar{u}_2}^2\right)^2\beta_{2i}\bar{u}_2}{\left(S_{z_2}^2\right)^2} + \frac{(\bar{z}_2+\bar{u}_2)S_{\bar{u}_2}^2\beta_{2i}}{S_{z_2}^2}\left(\sum_{j=1}^{n}\frac{z_{2j}\bar{u}_{2j.}}{n}\right) \\ -\frac{(\bar{z}_2+\bar{u}_2)\left(S_{\bar{u}_2}^2\right)^2\beta_{2i}}{\left(S_{z_2}^2\right)^2}\left(\sum_{j=1}^{n}\frac{z_{2j}\bar{u}_{2j.}}{n}\right) + \frac{(\bar{z}_2+\bar{u}_2)\left(S_{\bar{u}_2}^2\right)^2\beta_{2i}}{S_{z_2}^2} - \frac{(\bar{z}_2+\bar{u}_2)\left(S_{\bar{u}_2}^2\right)^3\beta_{2i}}{\left(S_{z_2}^2\right)^2} \end{array}\right]. \quad (41)$$

It can be shown that under the condition $u_{2jk} \sim N(0,\sigma_{u2j}^2)$, the following expressions are valid:

$$E(\bar{u}_2 S_{\bar{u}_2}^2) = 0; E(\bar{u}_2^2 S_{\bar{u}_2}^2) \sim O(1/n^2); E(\bar{u}_2^3 S_{\bar{u}_2}^2) \sim O(1/n^2); E(\bar{u}_2 S_{\bar{u}_2}^4) = 0; E(\bar{u}_2^2 S_{\bar{u}_2}^4) \sim O(1/n^2);$$

$$E(\bar{u}_2^3 S_{\bar{u}_2}^4) = 0; E(S_{\bar{u}_2}^2 \sum_{j=1}^{n} \bar{u}_{2j.}) = 0; E(S_{\bar{u}_2}^2 \bar{u}_2 \sum_{j=1}^{n} \bar{u}_{2j.}) = 0; E(S_{\bar{u}_2}^4 \bar{u}_2 \sum_{j=1}^{n} \bar{u}_{2j.}) \sim O(1/n^2);$$

$$E(S_{\bar{u}_2}^4) = 3/n + O(1/n^2); E(S_{\bar{u}_2}^6) \sim O(1/n^3); E(\bar{u}_2 S_{\bar{u}_2}^6) = 0.$$

Then, as $n \to \infty$, $E\left(\frac{Aq^{13}\hat{\beta}_{2i\_ols/hme}}{1-Aq^{33}}\right)$ in (41) approaches

$$E\left(\frac{Aq^{13}\hat{\beta}_{2i\_ols/hme}}{1-Aq^{33}}\right) \to -\frac{\beta_{2i}\bar{z}_2\sigma_{\bar{u}_2}^2}{S_{z_2}^2}. \quad (42)$$

Thus, from (38) and (42), it can be concluded that the bias of $\hat{\beta}_{0i\_cs}$ approaches zero as $n \to \infty$.

Similarly, it can be shown that the estimators $\hat{\beta}_{1i\_ols/hme}$ and $\hat{\beta}_{1i\_cs}$ are both unbiased. By following the same approach, the bias of $\hat{\beta}_{2i\_ols/hme}$ in (33) can be written as

$$Bias\ of\ \hat{\beta}_{2i\_ols/hme} \simeq -\frac{\beta_{2i}\sigma_{\bar{u}_2}^2}{S_{z_2}^2}\left(1-\frac{1}{n}\right) + \frac{\beta_{2i}}{\left(S_{z_2}^2\right)^2}\left(1-\frac{1}{n}\right)\left\{\frac{3}{n}+\frac{2}{n^2}E\left(\sum_{j=1}^{n}\sum_{l=j+1}^{n}\bar{u}_{2j.}^2\bar{u}_{2l.}^2\right)\right\}. \quad (43)$$

As $n \to \infty$, the bias of $\hat{\beta}_{2i\_ols/hme}$ in (43) approaches

$$Bias\ of\ \hat{\beta}_{2i\_ols/hme} \to -\frac{\beta_{2i}\sigma_{\bar{u}_2}^2}{S_{z_2}^2}. \quad (44)$$

The bias of $\hat{\beta}_{2i\_cs}$ in (35) can be written as $Bias\ of\ \hat{\beta}_{2i\_cs}$

$$\simeq Bias\ of\ \hat{\beta}_{2i\_ols/hme} + \frac{\beta_{2i}}{S_{z_2}^2}\left[\begin{array}{l} E\left(S_{\bar{u}_2}^2\right) + \frac{1}{S_{z_2}^2}E\left((\bar{z}_2+\bar{u}_2)\bar{u}_2 S_{\bar{u}_2}^2\right) - \frac{1}{\left(S_{z_2}^2\right)^2}E\left((\bar{z}_2+\bar{u}_2)\bar{u}_2\left(S_{\bar{u}_2}^2\right)^2\right) \\ -\frac{1}{nS_{z_2}^2}E\left(S_{\bar{u}_2}^2\sum_{j=1}^{n} z_{2j}\bar{u}_{2j.}\right) + \frac{1}{n\left(S_{z_2}^2\right)^2}E\left(\left(S_{\bar{u}_2}^2\right)^2\sum_{j=1}^{n} z_{2j}\bar{u}_{2j.}\right) \\ -\frac{1}{\left(S_{z_2}^2\right)^2}E\left(\left(S_{\bar{u}_2}^2\right)^2\right) + \frac{1}{\left(S_{z_2}^2\right)^2}E\left(\left(S_{\bar{u}_2}^2\right)^3\right) \end{array}\right]. \quad (45)$$

As $n \to \infty$, it can be shown that the bias of $\hat{\beta}_{2i\_cs}$ in (45) approaches zero.

This completes the proof of Theorem 1.