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Khatri-Rao sums for Hilbert space operators

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Abstract

We generalize the notions of Khatri-Rao sums for matrices and tensor sums for Hilbert space operators to Khatri-Rao sums for Hilbert space operators. This kind of operator sum is compatible with algebraic operations and order relations. We investigate its analytic properties, including continuity, convergence, and norm bounds. We also discuss the role of selection operator that relates Khatri-Rao sums to Tracy-Singh sums and Khatri-Rao products. Binomial theorem involving Khatri-Rao sums and its consequences are then established.

Keywords: tensor product, Khatri-Rao product (sum), Tracy-Singh product (sum), Moore-Penrose inverse, operator inequality

1. Introduction

In matrix and operator theory, there are many kinds of products which are of interest from both theory and application points of views. These products include the Kronecker (tensor) product, the Tracy-Singh product and the Khatri-Rao product. Let $M_n(\mathbb{C})$ denote the set of n-y-n complex matrices. Recall that for any $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$, the Kronecker product of A and B is given

$$A \hat{\otimes} B = \left[a_{ij} B \right]_{ij}. \tag{1}$$

The Tracy-Singh product of partitioned matrices, as a generalized Kronecker product, was introduced by Tracy and Singh (1972). Let $A = [A_{ij}]$ be partitioned with A_{ij} of order m_i x

m_j as the (i, j)th submatrix and let $B - [B_{kl}]$ be a partitioned matrix with B_{kl} of order n_k x n_l as the (i, j)th submatrix where $\sum_{i=1}^{r} m_i = m$, $\sum_{k=1}^{s} n_k = n$. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B = \left[\left[A_{ij} \hat{\otimes} B_{kl} \right]_{kl} \right]_{ij}. \tag{2}$$

If r and s are equal, the Khatri-Rao product of A and B can be defined to be (Khatri & Rao, 1968)

$$A \hat{\boxdot} B = \left[A_{ij} \hat{\otimes} B_{ij} \right]_{ij}. \tag{3}$$

See more information about the development of this theory in Al Zhour (2012), Al Zhour and Kilicman (2006b), Cao, Zhang and Yang (2002), Liu and Trenkler (2008), Shuangzhe (1999), Van Loan (2000).

From Tracy-Singh and Khatri-Rao products of matrices, Al Zhour and Kilicman (2006a) defined Tracy-Singh

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sums and Khatri-Rao sums of matrices and established certain properties for these sums. Indeed, the Tracy-Singh and Khatri-Rao sums are respectively defined to be

$$A \hat{\boxplus} B = A \hat{\boxtimes} I_n + I_m \hat{\boxtimes} B, \quad A \hat{\circledast} B = A \hat{\boxdot} I_n + I_m \hat{\boxdot} B \tag{4}$$

where I_m and I_n are block identity matrices of order m x m and n x n, respectively.

The theory of Kronecker products of matrices was extended to tensor products of bounded linear operators on a Hilbert space. The notion of tensor sum for operators was investigated in Kubrusly and Levan (2011). Indeed, the tensor sum of $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ is defined to be $A \otimes I_\mathcal{K} + I_\mathcal{H} \otimes B$, here $\otimes$ denotes the tensor product. Recently, the Tracy-Singh product and the Khatri-Rao product for operators were investigated by the authors (Ploymukda & Chansangiam, 2016).

From the discussion above, it is natural to generalize the Khatri-Rao sum for matrices and the tensor sum of operators to the Khatri-Rao sum for operators. We will show in this paper that the Khatri-Rao sum is compatible with algebraic operations and order relations. We discuss analytic properties of this sum, including continuity, convergence, norm bounds. We investigate the role of selection operator that relates Khatri-Rao sums to Tracy-Singh sums and Khatri-Rao products. Binomial theorem involving Khatri-Rao sums and its consequences are then established.

This paper is organized as follows. Section 2 supplies preliminary results on Tracy-Singh products and Khatri-Rao products for Hilbert space operators. In Section 3, we introduce Khatri-Rao sums for operators and investigate its algebraic, order, and analytic properties. Section 4 discusses relationship between Khatri-Rao sums and selection operators. In Section 5, we prove an operator version of binomial theorem and deduce some consequences inequalities.

2. Preliminaries

In what follows, $\mathcal{H}$ and $\mathcal{K}$ are complex separable Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, denote by

$\mathbb{B}(\mathcal{X},\mathcal{Y})$ the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X},\mathcal{X})$ to $\mathbb{B}(\mathcal{X})$. For Hermitian operators $A, B \in \mathbb{B}(\mathcal{H})$, the partial order $A \geqslant B$ indicates that $A - B$ is a positive operator, while $A > 0$ means that $\langle Ax, x \rangle > 0$ for all nonzero vectors x in $\mathcal{H}$.

2.1 Tracy-Singh products for operators

Recall that the tensor product of $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into itself such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \tag{5}$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. To define the Tracy-Sing product of operators, we first use projection theorem for Hilbert spaces to decompose

$$\mathcal{H} = \bigoplus_{i=1}^{m} \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{k=1}^{n} \mathcal{K}_k$$

where each $\mathcal{H}_I$ amd $\mathcal{K}_k$ are Hilbert spaces. Thus any operator $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ can be uniquely represented by operator matrices $A = \left[A_{ij} \right]_{i,j=1}^{m,m}$ and $B = \left[B_{kl} \right]_{k,l=1}^{n,n}$ where $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$ for each i, j, k, l..

Definition 1. According to the previous discussion, the *Tracy-Singh product* of A and B is defined to be the bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$ into itself represented by the operator matrix

$$A \boxtimes B = \left[\left[A_{ij} \otimes B_{kl} \right]_{kl} \right]_{ij}. \tag{6}$$

Lemma 2. (Ploymukda & Chansangiam, 2016)**.** The following properties hold, provided that all operators are compatible.

$$(A \boxtimes B)^* = A^* \boxtimes B^*, \tag{7}$$

$$(\alpha A) \boxtimes B = \alpha(A \boxtimes B) = A \boxtimes (\alpha B), \tag{8}$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \tag{9}$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A, \tag{10}$$

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD. \tag{11}$$

Lemma 3. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. Then $A \boxtimes B = 0$ if and only if $A = 0$ or $B = 0$.

Proof. Write $A = [A_{ij}]_{i,j=1}^{m,m}$ and $B = [B_{kl}]_{k,l=1}^{n,n}$. We have the following norm bounds (Ploymukda & Chansangiam, 2016).

$$\frac{1}{mn}\|A\|\|B\| \leqslant \|A \boxtimes B\| \leqslant mn\|A\|\|B\|.$$

Now, the claim follows.

Definition 4. Let $A = \left[A_{ij}\right]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{kl}\right]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$. We define the *Tracy Singh sum* of A and B as follows:

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B \tag{12}$$

which belongs to $\mathbb{B}\left(\bigoplus_{i,j=1}^{m,n}\mathcal{H}_i \otimes \mathcal{K}_j\right)$.

2.2 Khatri-Rao products for operators

From now on, fix the following decompositions of Hilbert spaces:

$$\mathcal{H} = \bigoplus_{i=1}^{n}\mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{i=1}^{n}\mathcal{K}_i.$$

Definition 5. Let $A = \left[A_{ij}\right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{ij}\right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ be operator matrices.

$$A \square B = \left[A_{ij} \otimes B_{ij}\right]_{i,j=1}^{n,n} \tag{13}$$

Which is bounded linear operator from $\bigoplus_{i=1}^{n}\mathcal{H}_i \otimes \mathcal{K}_i$ in to itself.

Recall that for each $A \in \mathbb{M}_n(\mathbb{C})$, the induced map $L_A : \mathbb{C}^m \to \mathbb{C}^m$, $x \mapsto Ax$ is a bounded linear operator.

Lemma 6. (Ploymukda & Chansangiam, 2016). For any complex matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ partitioned in block-matrix forms, we have

$$L_A \square L_B = L_{A\dot\square B}. \tag{14}$$

Lemma 7. (Ploymukda & Chansangiam, 2016). Let $A \in \mathbb{B}(\mathcal{H})$ and $B, C \in \mathbb{B}(\mathcal{K})$ be operator matrices, and let $\alpha \in \mathbb{C}$. Then

$$(A \square B)^* = A^* \square B^*, \tag{15}$$

$$(\alpha A) \square B = \alpha(A \square B) = A \square (\alpha B), \tag{16}$$

$$A \square (B + C) = A \square B + A \square C, \tag{17}$$

$$(B + C) \square A = B \square A + C \square A. \tag{18}$$

Lemma 8. (Ploymukda & Chansangiam, 2016). There is an isometry Z such that $ZZ^* \leqslant I$ and

$$A \square B = Z^*(A \boxtimes B)Z \tag{19}$$

for any $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. We call Z the selection operator associated with the ordered tuple $(\mathcal{H}, \mathcal{K})$.

Lemma 9. (Ploymukda & Chansangiam, 2016). Let Z be the selection operator associated with $(\mathcal{H}, \mathcal{K})$. Then there exists an operator.

$$W : \bigoplus_{i=1}^{n-1}\bigoplus_{j=1}^{n}\mathcal{H}_i \otimes \mathcal{K}_j \to \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n}\mathcal{H}_i \otimes \mathcal{K}_j$$

Such that $Z^*W = 0$ and $ZZ^* + WW^* = I$. Indeed, W is defined as

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} \tag{20}$$

where

$$W^{(s)} = \left[W_{kl}^{(s)}\right]_{k,l=1}^{n,n^2-1} : \bigoplus_{i=1}^{n}\bigoplus_{\substack{j=1 \\ i+j<n^2}}^{n}\mathcal{H}_i \otimes \mathcal{K}_i \to \bigoplus_{i=1}^{n}\mathcal{H}_s \otimes \mathcal{K}_i \text{ for } s = 1,...,n, \text{ with } W_{kl}^{(s)}$$

is an identity operator if $k \neq s$ and $l = n(s - 1) + k$ and others are zero operators.

Lemma 10. (Ploymukda & Chansangiam, 2016). Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices.

(i) If $A \geqslant 0$ and $B \geqslant 0$, then $A \square B \geqslant 0$.

(ii) If $A > 0$ and $B > 0$, then $A \square B > 0$.

Lemma 11. (Ploymukda & Chansangiam, 2016)**.** Let A $\in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices and let $\left(A_r\right)_{r=1}^{\infty}$ and $\left(B_r\right)_{r=1}^{\infty}$ be sequence in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$ respectively. If A_r $\to A$ and $B_r \to B$, then $A_r \,\square\, B_r \to A \,\square\, B$.

Lemma 12. (Ploymukda & Chansangiam, 2016)**.** For any operator matrices $A = \left[A_{ij}\right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{ij}\right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$, we have

$$\| A \,\square\, B \| \leqslant n^2 \, \| A \| \| B \|. \tag{21}$$

3. Algebraic, Order, and Analytic Properties of Khatri-Rao Sums for Operators

In this section, we generalize the Khatri-Rao sum for matrices and the tensor sum for operators to the Khatri-Rao sum for operators. This kind of operator sum turns out to be compatible with algebraic operations and order relations for operators. Binomial theorem involving Tracy-Singh sums and its consequences are also established. We investigate continuity, convergence, norm bounds for Tracy-Singh sums of operators.

Definition 13. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. We define the *Khatri-Rao sum* of A and B as follows:

$$A \circledast B = A \,\square\, I_{\mathcal{K}} + I_{\mathcal{H}} \,\square\, B \tag{22}$$

which belonging to $\mathbb{B}(\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i)$.

Note that if both A and B are 1 x 1 block operator matrices, their Khatri-Rao sum $A \circledast B$ becomes the tensor sum (Kubrusly & Levan, 2011)

$$A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B.$$

If $\mathcal{H}_i = \mathcal{K}_i = \mathbb{C}$, the Khatri-Rao sum $A \circledast B$ reduces to the Hadamard sum of complex matrices (Al Zhour & Kilicman, 2006a).

Proposition 14. Let $A = [A_{ij}]$ and $B = [B_{ij}]$ be compatible partitioned matrices of order of m x m and n x n, respectively. Then

$$L_A \circledast L_B = L_{A \hat{\oplus} B}. \tag{23}$$

Proof. We know that the linear map induced by the identity matrix is the identity operator. By applying Lemma 6, we get

$$L_A \circledast L_B = L_A \,\square\, L_I + L_I \,\square\, L_B = L_{A \hat{\square} I} + L_{I \hat{\square} B} = L_{A \hat{\oplus} B}. \tag{24}$$

Proposition 15. Let α and β be scalar. The following properties hold, provided that all operators are compatible.

$$\left(A \circledast B\right)^* = A^* \circledast B^*, \tag{24}$$

$$\alpha \left(A \circledast B\right) = \alpha A \circledast \alpha B, \tag{25}$$

$$\left(\alpha + \beta\right)\left(A \circledast B\right) = \alpha A \circledast \beta B + \beta A \circledast \alpha B, \tag{26}$$

$$\left(A_1 + A_2\right) \circledast \left(B_1 + B_2\right) = A_1 \circledast B_1 + A_2 \circledast B_2, \tag{27}$$

$$\left(A_1 \oplus A_2\right) \circledast \left(B_1 \oplus B_2\right) = \left(A_1 \circledast B_1\right) \oplus \left(A_2 \circledast B_2\right). \tag{28}$$

Here the symbol $\oplus$ denotes the direct sum for operators. We call property (27) mixed sum property.

Proof. Using Lemma 7, we get (24)-(26). By applying properties (17) and (18) Lemma 7, we get

$$\begin{aligned} \left(A+C\right) \circledast \left(B+D\right) &= \left(A+C\right) \,\square\, I + I \,\square\, \left(B+D\right) \\ &= A \,\square\, I + C \,\square\, I + I \,\square\, B + I \,\square\, D \\ &= \left(A \,\square\, I + I \,\square\, B\right) + \left(C \,\square\, I + I \,\square\, D\right) \\ &= A \circledast B + C \circledast D. \end{aligned}$$

By the definition of Khatri-Rao sum, we have

$$\begin{aligned} \left(A_1 \circledast B_1\right) \oplus \left(A_2 \circledast B_2\right) &= \begin{bmatrix} A_1 \,\square\, I + I \,\square\, B_1 & 0 \\ 0 & A_2 \,\square\, I + I \,\square\, B_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 \,\square\, I & 0 \\ 0 & A_2 \,\square\, I \end{bmatrix} + \begin{bmatrix} I \,\square\, B_1 & 0 \\ 0 & I \,\square\, B_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \,\square\, \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \,\square\, \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \\ &= \left(A_1 \oplus A_2\right) \circledast \left(B_1 \oplus B_2\right). \end{aligned}$$

Corollary 16. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are Hermitian (skew-Hermitian), then $A \circledast B$ is also Hermitian (skew-Hermitian).

Corollary 17. Let $A_1 \in \mathbb{B}(\mathcal{H})$ and $A_2 \in \mathbb{B}(\mathcal{K})$. If $A_1 = X_1 + iY_1$ and $A_2 = X_2 + iY_2$ are the Cartesian decompositions of A_1 and A_2 respectively, then $A_1 \circledast A_2 = X_1 \circledast X_2 + i(Y_1 \circledast Y_2)$ is the Cartesian decomposition of $A_1 \circledast A_2$.

Proposition 18. Let $A, A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B, B_1, B_2 \in \mathbb{B}(\mathcal{K})$ be operator matrices.

(i) If $A \geqslant 0$ and $B \geqslant 0$, then $A \circledast B \geqslant 0$.

(ii) If $A > 0$ and $B > 0$, then $A \circledast B > 0$.

(iii) If $A_1 \geqslant A_2$ and $B_1 \geqslant B_2$, then $A_1 \circledast B_1 \geqslant A_2 \circledast B_2$.

(iv) If $A_1 > A_2$ and $B_1 > B_2$, then $A_1 \circledast B_1 > A_2 \circledast B_2$.

Proof. It follows from Lemma 10 and Proposition 15.

The next proposition asserts that the Khatri-Rao sum is (jointly) continuous with respect to the operator-norm topology.

Proposition 19. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operators matrices, and let $\left(A_r \right)_{r=1}^{\infty}$ and $\left(B_r \right)_{r=1}^{\infty}$ be sequence in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$ respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \circledast B_r \to A \circledast B$.

Proof. By Lemma 11, we have $A_r \square I \to A \square I$ and $I \square B_r \to I \square B$. Then $A_r \square I + I \square B_r \to A \square I + I \square B$, i.e. $A_r \circledast B_r \to A \circledast B$.

The next result is a triangle-like inequality involving Khatri-Rao sums.

Proposition 20. For any operator matrices $A = \left[A_{ij} \right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{ij} \right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$, we have

$$\frac{1}{n^2} \| A \circledast B \| \leqslant \| A \| + \| B \|. \tag{29}$$

Proof. By using Lamma 12, We have

$$\| A \circledast B \| = \| A \square I + I \square B \| \leqslant \| A \square I \| + \| I \square B \|$$

$$\leqslant n^2 \| A \| \| I \| + n^2 \| I \| \| B \| = n^2 (\| A \| + \| B \|).$$

4. Khatri-Rao Sums and Selection Operators

Proposition 21. There is an isometry Z such that $ZZ^* \leqslant I$ and

$$A \circledast B = Z^* \left(A \boxplus B \right) Z \tag{30}$$

for any $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$.

Proof. By Lemma 8, there is an isometry Z such that $ZZ^* \leqslant I$ and

$$A \circledast B = A \square I + I \square B = Z^* (A \boxtimes I + I \boxtimes B) Z = Z^* (A \boxplus B) Z.$$

Proposition 22. Let $A_1 \in \mathbb{B}(\mathcal{H})$ and $A_2 \in \mathbb{B}(\mathcal{K})$ and let Z be the selection operator associated with $(\mathcal{H}, \mathcal{K})$. Then

$$A \square B^* + A^* \square B \leqslant (A \circledast B)(A \circledast B)^* + Z^* (A \boxplus B) WW^* (A \boxplus B)^* Z. \tag{31}$$

where W is the operator defined in Lemma 9.

Proof. By Lemmas 2, 8 and 9, and Theorem 21, we have

$$(A \circledast B)(A \circledast B)^* = Z^* (A \boxplus B) ZZ^* (A^* \boxplus B^*) Z$$

$$= Z^* (A \boxplus B)(I - WW^*)(A^* \boxplus B^*) Z$$

$$= Z^* (A \boxplus B)(A^* \boxplus B^*) Z - Z^* (A \boxplus B) WW^* (A^* \boxplus B^*) Z$$

$$= Z^* (AA^* \boxplus BB^* + A \boxtimes B^* + A^* \boxtimes B) Z$$

$$- Z^* (A \boxplus B) WW^* (A^* \boxplus B^*) Z$$

$$= AA^* \circledast BB^* + A \square B^* + A^* \square B - Z^* (A \boxplus B) WW^* (A \boxplus B)^* Z.$$

By Proposition 18, we have $AA^* \circledast BB^* \geqslant 0$ and hence

$$(A \circledast B)(A \circledast B)^* + Z^* (A \boxplus B) WW^* (A \boxplus B)^* Z - (A \square B^* + A^* \square B) \geqslant 0.$$

Proposition 23. Let $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$ be positive operators. Then

$$AC \circledast BD \;\leqslant\; (A \circledast B)(C \circledast D) + Z^{*}(A \boxplus B)WW^{*}(C \boxplus D)Z. \tag{32}$$

Proof. By replacing $A^{*} = C$ and $B^{*} = D$ in the proof of Proposition 22, we have

$$(A \circledast B)(C \circledast D) + Z^{*}(A \boxplus B)WW^{*}(C \boxplus D)Z - AC \circledast BD = A \square D + C \square B.$$

Since $A \square D + C \square B$ is positive, we get the result.

5. Binomial Theorem Involving Khatri-Rao Sums and Its Consequences

In this section, we prove an operator version of binomial theorem concerning Khatri- Rao sums. As consequences, we obtain two operator inequalities, including Bernoulli type inequality.

Lemma 24. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. We have $W^{*}(A \boxplus B)Z = 0$ if and only if $A = A_{1} \oplus \ldots \oplus A_{n}$ and $B = B_{1} \oplus \ldots \oplus B_{n}$ where Z is the selection operator associated with $(\mathcal{H}, \mathcal{K})$ and W is the operator defined by (20).

Proof. Consider $A \in \mathbb{B}(\mathcal{H}_{1} \oplus \mathcal{H}_{2})$ and $B \in \mathbb{B}(\mathcal{K}_{1} \oplus \mathcal{K}_{2})$. By a computation, we have

$$W^{*}(A \boxplus B)Z \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}^{*} \begin{bmatrix} A_{11} \oplus B_{11} & I \otimes B_{12} & A_{12} \otimes I & 0 \\ I \otimes B_{21} & A_{11} \oplus B_{21} & 0 & A_{12} \otimes I \\ A_{21} \otimes I & 0 & A_{22} \oplus B11 & I \otimes B_{12} \\ 0 & A_{21} \otimes I & I \otimes B_{21} & A_{22} \oplus B_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}$$

$$=\; \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} A_{11} \oplus B_{11} & 0 \\ I \otimes B_{21} & A_{12} \otimes I \\ A_{21} \otimes I & I \otimes B21 \\ 0 & A_{22} \oplus B22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I \otimes B_{21} & A_{12} \otimes I \\ A_{21} \otimes I & I \otimes B_{12} \end{bmatrix}.$$

By Lemma 3, we obtain

$$W^{*}(A \boxplus B)Z \;=\; 0 \quad \Leftrightarrow \quad A_{12} \otimes I, A_{21} \otimes I, I \otimes B_{12} \text{ and } I \otimes B_{21} \text{ are zeros.}$$

$$\Leftrightarrow \quad A_{12}, A_{21}, B_{12} \text{ and } B_{21} \text{ are zeros.}$$

$$\Leftrightarrow \quad A = A_{1} \oplus A_{2} \text{ and } B = B_{1} \oplus B_{2}.$$

A direct computation shows that Lemma 24 holds for the case $A = [A_{ij}]_{i,j=1}^{n,n}$ and $B = [B_{ij}]_{i,j=1}^{n,n}$ for any integer $n > 2$.

Theorem 25. Let $A = A_{1} \oplus \ldots \oplus A_{n} \in \mathbb{B}(\mathcal{H})$ and $B = B_{1} \oplus \ldots \oplus B_{n} \in \mathbb{B}(\mathcal{K})$ be compatible operator matrices, then for any integer $r \geqslant 2$,

$$(A \circledast B)^{r} \;=\; A^{r}B^{r} + \sum_{k=1}^{r-1} \binom{r}{k}\left(A^{r-k} \square B^{k}\right). \tag{33}$$

Proof. By the proof of Proposition 22, we have

$$(A \circledast B)^{2} \;=\; A^{2} \circledast B^{2} + 2A \square B - Z^{*}(A \boxplus B)WW^{*}(A \boxplus B)Z.$$

Since $W^{*}(A \boxplus B)Z = 0$, by Lemma 24, we get

$$(A \circledast B)^{2} \;=\; A^{2} \circledast B^{2} + 2A \square B.$$

This show that Theorem 25 is true for $r = 2$. Suppose that (33) hold for $r \geqslant 2$. Then, by Lemma 2,

$$(A \circledast B)^{r+1} \;=\; (AB)^{r}(AB)$$

$$=\; \left(A^{r}B^{r} + \sum_{k=1}^{r-1}\binom{r}{k}\left(A^{r-k} \square B^{k}\right)\right)(AB)$$

$$=\; (A^{r}B^{r})(AB) + \left(\sum_{k=1}^{r-1}\binom{r}{k}\left(A^{r-k} \square B^{k}\right)\right)(AB)$$

$$=\; A^{r+1}B^{r+1} + A^{r} \square B + A \square B^{r} - Z^{*}(A^{r} \boxplus B^{r})WW^{*}(A \boxplus B)Z$$

$$+ \sum_{k=1}^{r-1}\binom{r}{k}\left(A^{r-k+1} \square B^{k} - Z^{*}(A^{r-k} \boxtimes B^{r})WW^{*}(A \boxtimes I)Z\right)$$

$$+ \sum_{k=1}^{r-1}\binom{r}{k}\left(A^{r-k} \square B^{k+1} - Z^{*}(A^{r-k} \boxtimes B^{r})WW^{*}(I \boxtimes B)Z\right)$$

$$=\; A^{r+1}B^{r+1} + A^{r} \square B + A \square B^{r} - Z^{*}(A^{r} \boxplus B^{r})WW^{*}(A \boxplus B)Z$$

$$+ \sum_{k=1}^{r}\binom{r}{k}\left(A^{(r+1)-k} \square B^{k}\right) + \sum_{k=0}^{r-1}\binom{r}{k}\left(A^{r-k} \square B^{k+1}\right)$$

$$- \sum_{k=1}^{r-1}\binom{r}{k}Z^{*}(A^{r-k} \boxtimes B^{r})WW^{*}(A \boxplus B)Z$$

$$=\; A^{r+1}B^{r+1} + A^{r} \square B + A \square B^{r} - Z^{*}(A^{r} \boxplus B^{r})WW^{*}(A \boxplus B)Z$$

$$+ \sum_{k=1}^{r}\binom{r}{k}\left(A^{(r+1)-k} \square B^{k}\right) - \sum_{k=1}^{r-1}\binom{r}{k}Z^{*}(A^{r-k} \boxtimes B^{r})WW^{*}(A \boxplus B)Z.$$

Since $W^{*}(A \boxplus B)Z = 0$ (by Lemma 24), we arrive at (33).

Corollary 26. Let $A = A_{1} \oplus \ldots \oplus A_{n} \in \mathbb{B}(\mathcal{H})$ and $B = B_{1} \oplus \ldots \oplus B_{n} \in \mathbb{B}(\mathcal{K})$ be positive operators, then for any $r \in \mathbb{N}$,

$$(A \circledast B)^{r} \;\geqslant\; A^{r} \circledast B^{r}. \tag{34}$$

Proof. Inequality (34) is obviously true for $r = 1$. Let $r \geqslant 2$. Since $A, B \geqslant 0$, we have by Lemma 10 that $\sum_{k=1}^{r}\binom{r}{k}\left(A^{r-k} \square B^{k}\right) \geqslant 0$. By Theorem 25, we obtain

$$(A \circledast B)^{r} \;\geqslant\; A^{r} \circledast B^{r}.$$

Corollary 27. Let $A = A_{1} \oplus \cdots \oplus A_{n} \in \mathbb{B}(\mathcal{H})$ be a positive operator. Then for any $r \in \mathbb{N}$,

$$(I \circledast A)^{r} \;\geqslant\; I \circledast (rA). \tag{35}$$

Proof. Since $A \geqslant 0$ we have $I \circledast A^{r} \geqslant I \square I$. By Theorem 26, we obtain

$$(I \circledast A)^{r} \;=\; I \circledast A^{r} + \sum_{k=1}^{r-1}\binom{r}{k}\left(I \square A^{k}\right) = I \circledast A^{r} + r(I \square A) + \sum_{k=2}^{r-1}\binom{r}{k}\left(I \square A^{k}\right)$$

$$\geqslant\; I \circledast A^{r} + r(I \square A) \;\geqslant\; I \square I + I \square (rA) \;=\; I \circledast (rA).$$

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