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Original Article

Fuzzy Directed Divergence Measure and Its Application to Decision Making

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Abstract

Divergence or relative information is a measure of information associated with two probability distributions of a discrete random variable which is based in Shannon entropy. In this paper a new divergence measure and corresponding fuzzy directed divergence measure have been proposed. Comparative study of the proposed divergence measure with some existing divergence measure has been done with the help of numerical example. Further, the application of proposed fuzzy directed divergence is illustrated in decision making problems.

Keywords: fuzzy entropy, fuzzy divergence, fuzzy directed divergence

1. Introduction

Information theory deals with the study of problems concerning any system that includes information dispensation, storage, retrieval and decision making. In other words, information theory studies all problems related to the entity called communication system. The source of messages can be a person or machine that generates the messages, the encoder converts messages in to an object which is suitable for transmission, such as a sequence of binary digits (digital computer applications), channel is a medium over which the coded message is transmitted, decoder convert the received output from the channel and tries to convert the received output in to the original message to transport it to the destination. But this cannot be done with absolute consistency due to existence of some disorder in the system, which is also termed as noise. Information theory is considered to be identified by Shannon (1948); measure of information theory is termed as entropy w.r.t. probability distribution. Shannon (1948) also proved various mathematical properties of the measure. Kullback and Liebler (1952) quantified the measure of information associated with the two probability distributions $p = (p_1, p_2, \dots \dots, p_n)$ and $q = (q_1, q_2, \dots \dots, q_n)$ of discrete random variables, $D(p \parallel q) = \sum_{i=1}^{n} p_i log\frac{p_i}{q_i}$ known as directed divergence. There exist other measures of divergence on set of probabilities, with diverse names such as distance and discrimination etc. The natural properties of directed divergence are divergence is a non-negative function; it becomes zero when two sets coincide.

Analogous to probability theory, fuzzy set theory was introduced by Zadeh (1965). Uncertainty and fuzziness are present in human thinking and related to many practical problems. Fuzziness is found in our language, in our judgment and in the course of actions.

Fuzzy set theory gave a whole new dimension to set theory which considers an element to belong to a set or does not belong to a set. A fuzzy set $\tilde{A}$ is subset of universe of discourse X, is defined as $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle / x \in X\}$, where $\mu_{\tilde{A}} : X \to [0,1]$ is a membership function of $\tilde{A}$. The value of $\mu_{\tilde{A}}(x)$ describes the degree of belongingness of $x \in X$ in $\tilde{A}$.

Fuzzy entropy deals with vagueness and ambiguous uncertainties, whereas Shannon entropy deals with probabilistic uncertainties. De Luca and Termini (1972) characterized the fuzzy entropy and introduced a set of following

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properties (1–4) for which fuzzy entropy should satisfy them:

1. Fuzzy entropy is minimum iff set is crisp.
2. Fuzzy entropy is maximum when membership value is 0.5.
3. Fuzzy entropy decreases if set is sharpened.
4. Fuzzy entropy of a set is same as its complement.

Bhandari and Pal (1993) gave a fuzzy information measure for discrimination of a fuzzy set $\tilde{A}$ relative to some other fuzzy set $\tilde{B}$ called as Fuzzy divergence and gave various measures of fuzzy entropy and measure of fuzzy divergence corresponding to a fuzzy set $\tilde{A}$ relative to some other fuzzy set $\tilde{B}$. Let X be a Universal set and F(X) be the collection of all fuzzy subsets of X. A mapping D: F(X)×F(X) →R is called divergence between two fuzzy subsets if it satisfies following properties for any $\tilde{A}, \tilde{B}, \tilde{C} \in F(X)$:

1. D $(\tilde{A}, \tilde{B})$ is non-negative.
2. D$(\tilde{A}, \tilde{B})$= D$(\tilde{B}, \tilde{A})$
3. D$(\tilde{A}, \tilde{B}) = 0$ if $\tilde{A} = \tilde{B}$
4. Max{D$(\tilde{A} \cup \tilde{C}, \tilde{B} \cup \tilde{C})$, D$(\tilde{A} \cap \tilde{C}, \tilde{B} \cap \tilde{C})$}≤ D$(\tilde{A}, \tilde{B})$

The simplest fuzzy directed divergence is $D(A,B) = \sum_{i=1}^{n}\left[\mu_A(x_i) log\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i)) log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right]$ given by Bhandari and Pal (1993), where $\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)$ describes the degree of belongingness of $x_i \in X$ in A and $\mu_B(x_1), \mu_B(x_2), \dots\dots\dots, \mu_B(x_n)$ describes the degree of belongingness of $x_i \in X$ in B respectively. Later, Fan and Xie (1999) gave the discrimination of fuzzy information of fuzzy set A against B.

$$I(A,B) = \sum_{i=1}^{n}\left[1 - (1-\mu_A(x_i))e^{\mu_A(x_i)-\mu_B(x_i)} - \mu_A(x_i)e^{(\mu_B(x_i)-\mu_A(x_i))}\right]$$

w.r.t. exponential fuzzy entropy given by Pal and Pal (1989). Further, corresponding to entropy given by Havrda-Charvat (1967), Kapur (1997) gave a generalized measure of fuzzy directed divergence as $I_\alpha(A,B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}[\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha} - 1]$, $\alpha \neq 1, \alpha > 0$ Divergence is found in various applications in the real world such as image segmentation, medical sciences, pattern recognition, fuzzy clustering etc.

Operations on fuzzy sets are termed as aggregation operators such as fuzzy union and fuzzy intersection. These operations are used to combine two or more fuzzy sets into one. An aggregation operation (Klir & Folger, 1988) defined as a function $A: [0,1]^n \to [0,1]$ satisfying:

1. A(0,0,…..0)=0 and A(1,1,…..,1)=1
2. A is monotonic in each argument.

Bhatia and Singh (2013), proposes a measure of arithmetic–geometric directed divergence of two arbitrary fuzzy sets A and B is as

$$T(A,B) = \sum_{i=1}^{n}\left[\frac{\mu_A(x_i)+\mu_B(x_i)}{2} log\frac{\mu_A(x_i)+\mu_B(x_i)}{2\sqrt{\mu_A(x_i)\,\mu_B(x_i)}} + \frac{2-\mu_A(x_i)-\mu_B(x_i)}{2} log\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}\right]$$

and defined generalized triangular discrimination between two arbitrary fuzzy sets A and B as follows:

$$\Delta_\alpha(A,B) = \sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^{2\alpha}\left[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))^{2\alpha-1}} + \frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))^{2\alpha-1}}\right]$$

They also defined a new (α, β) class of measure of fuzzy directed divergence for two arbitrary sets A and B as

$$D_\alpha^\beta(A,B) = \frac{1}{\beta-1}\sum_{i=1}^{n}\left[\left(\mu_A(x_i)^\alpha\mu_B(x_i)^{1-\beta} + (1-\mu_A(x_i))^{1-\alpha}(1-\mu_B(x_i))^{1-\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right]$$

$\alpha > 0, \alpha \neq 1, \beta > 0, \beta \neq 1$ and introduced (α, β) generalized arithmetic-geometric measure of fuzzy directed divergence $T_\alpha^\beta(A,B) = \frac{1}{2}\left[D_\alpha^\beta\left(\frac{A+B}{2}, A\right) + D_\alpha^\beta\left(\frac{A+B}{2}, B\right)\right]$ and $T_\alpha^\beta(A,B) = T(A,B) at\ \alpha = \beta = 1$.

A new measure of fuzzy directed divergence for two Fuzzy sets A and B,

$$M_{H^* \quad A^*}^F(A,B) = \sum_{i=1}^{n}\frac{(\mu_A(x_i)-\mu_B(x_i))^2}{2}\left[\frac{1}{\mu_A(x_i)-\mu_B(x_i)} + \frac{1}{2-\mu_A(x_i)-\mu_B(x_i)}\right]$$

Where, $A^*: [0,1]^2 \to [0,1]$ such that $A^*(a,b) = \frac{a+b}{2}$ and $H^*: [0,1]^2 \to [0,1]$ such that $H^*(a,b) = \frac{a^2+b^2}{a+b}$ was defined by Bhatia and Singh (2013), they also discussed application of new directed divergence measure in images segmentation.

Bhatia and Singh (2013), introduced three new divergence measures between fuzzy sets and some properties of these divergence measures. They also defined three aggregation functions corresponding to divergence measures.

Verma et al. (2012), defined a measure of entropy as

$$V_a(P) = \sum_{i=1}^{n}\ln(1+ap_i) - \sum_{i=1}^{n} lnp_i - \ln(1+a),\ a > 0$$

for probability distribution, $P = (p_1, p_2, \dots\dots, p_n)$ and its corresponding measure of directed divergence is defined as

$$D_a(P:Q) = \sum_{i=1}^{n} q_i ln\frac{p_i}{q_i} - \sum_{i=1}^{n} q_i \ln\left(\frac{q_i+ap_i}{q_i}\right) + \ln(1+a), a > 0$$

and corresponding measure of fuzzy directed divergence is

$$D(A,B) = \sum_{i=1}^{n}\mu_B(x_i) ln\left(\frac{\mu_A(x_i)}{a\mu_A(x_i)+\mu_B(x_i)}\right) + \sum_{i=1}^{n}(1-\mu_B(x_i))\, ln\left(\frac{1-\mu_A(x_i)}{1+a-a\mu_A(x_i)-\mu_B(x_i)}\right) + \ln(1+a),\quad a > 0$$

and their properties were studied.

Li *et al*. (2014a) proposed two approaches to define divergence measure under fuzzy settings based on dissimilarity functions and fuzzy equivalences. Li *et al.* (2014b) proposed different methods to define fuzzy equivalences and used it to define similarity measures for fuzzy sets. Tomar and Ohlan (2014a) present various fuzzy mean divergence measures and inequalities amongst them. Tomar and Ohlan (2014b) introduce a parametric generalized exponential measure of fuzzy divergence of order α and established a relationship between entropy of order α and proposed divergence measure. Tomar and Ohlan (2015) proposed a generalized measure of fuzzy divergence and applied it to multi-criteria decision making problems. Li *et al*. (2016) discussed the robustness of fuzzy connectives and reasoning using general divergence measure. He *et al*. (2016) proved the T_L transitivity is satisfied by the similarity measures defined by Li *et al*. (2014) and also investigated its fuzzy equivalences.

In the next section, we have proposed two new binary aggregation operators and corresponding to these operators a new divergence measure has been introduced. Further, a new directed divergence measure for two fuzzy sets have been defined.

2. Fuzzy Directed Divergence

Aggregation operators are defined as $G:[0,1]^2 \rightarrow [0,1]$, $G(a,b)=ab$ and $K:[0,1]^2 \rightarrow [0,1]$, $K(a,b)=\frac{(a+b)^4}{8(a^2+b^2)}$, both the functions are monotonic in each argument and satisfy boundary conditions and hence are aggregation operators. Let us define a new divergence as

$$D_{G,K}(P,Q)=\sum_{i=1}^{n}\left[\frac{(p_i+q_i)^4}{8(p_i^2+q_i^2)}-p_iq_i\right] \quad (1)$$

$$=\sum_{i=1}^{n} D_{G,K}(p_i,q_i)$$

where $P=(p_1,p_2,\dots\dots,p_n)$ and $Q=(q_1,q_2,\dots\dots,q_n)$

Equation 1 is clearly non-negative as shown graphically in Figure 1 (where $a=p_i, b=q_i$ and $z=D_{G,K}(p_i,q_i)$).

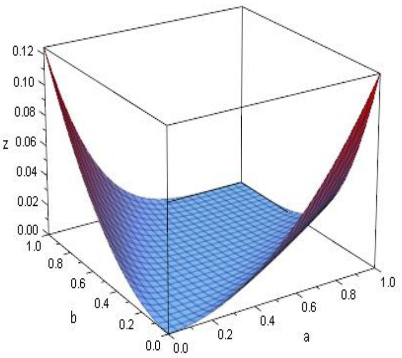

Figure 1. $D_{G,K}(p_i,q_i)$

Hence $D_{G,K}(P,Q)$ is a non-negative function with minimum value $D_{G,K}(P,Q)=0$ as $P=Q$ and

$$\frac{\partial D_{G,K}(P,Q)}{\partial p_i}=\sum_{i=1}^{n}\left[\frac{4(p_i+q_i)^3}{8(p_i^2+q_i^2)}-q_i-\frac{16p_i(p_i+q_i)^4}{(8(p_i^2+q_i^2))^2}\right] \quad (2)$$

Equation 2 is zero when each $p_i=0=q_i$ and

$$\frac{\partial^2 D_{G,K}(P,Q)}{\partial {p_i}^2}=\sum_{i=1}^{n}\left[\begin{array}{c}\frac{12(p_i+q_i)^2}{8(p_i^2+q_i^2)}-\frac{16(p_i+q_i)^4}{(8(p_i^2+q_i^2))^2}\\ -\frac{128p_i(p_i+q_i)^3}{(8(p_i^2+q_i^2))^2}+\frac{512(p_i^2(p_i+q_i)^4}{(8(p_i^2+q_i^2))^4}\end{array}\right]$$

$$=\sum_{i=1}^{n} D''_{G,K}(p_i,q_i)$$

Figure 2 (where $a=p_i$, $b=q_i$ and $z=D'_{G,K}(p_i,q_i)$) graphically represents $D'_{G,K}(p_i,q_i)$, $0\le p_i,q_i\le 1$ and shows that $\frac{\partial^2 D_{G,K}(P,Q)}{\partial {p_i}^2}\ge 0$ in $0\le p_i,q_i\le 1$.

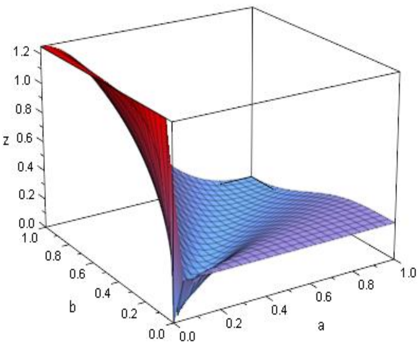

Figure 2. $\frac{\partial^2 D_{G,K}(P,Q)}{\partial {p_i}^2}$

$D_{G,K}(P,Q)$ is a convex function for $0\le p_i,q_i\le 1$. Hence eq. (1) is a valid measure of divergence.

Now, compare the membership values by comparing the fuzziness of A, B with the fuzziness of the intermediate subset with the help of an example. Consider the universe consisting of four elements A, B, C and D with membership values given as follows:

Membership Values

	x_1	x_2	x_3	x_4
A	0.25	0.8	0.5	0.3
B	0.2	0.7	0.5	0.2
C	0.04	0.5	0.7	0.5
D	0.89	0.01	0.1	0.99

For these sets we obtain $D_{G,K}(A,B)=0.000115$, $D_{G,K}(A,C)=0.005789$ and $D_{G,K}(A,D)=0.139388$. The divergence measure shows that A is quite similar to B and different from D.

Measure of fuzzy directed divergence between two fuzzy sets A and B corresponding to (2) is defined as

$$F_{G,K}(A,B) = \sum_{i=1}^{n}\left[\begin{array}{c}\frac{(\mu_A(x_i)+\mu_B(x_i))^4}{8(\mu_A(x_i)^2+\mu_B(x_i)^2)}-\mu_A(x_i)\mu_B(x_i)\\ +\frac{(2-\mu_A(x_i)-\mu_B(x_i))^4}{8((1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2)}-(1-\mu_A(x_i))(1-\mu_B(x_i))\end{array}\right]$$

$$= \sum_{i=1}^{n} F_{G,K}^{i}(A,B) \tag{3}$$

Clearly, $F_{G,K}(A,B)$ is non-negative, $F_{G,K}(A,B) = F_{G,K}(B,A)$ and $F_{G,K}(A,A) = 0$. To prove $\max\{F_{G,K}(A \cup C, B \cup C), F_{G,K}(A \cap C, B \cap C)\} \le F_{G,K}(A,B)$, we divide the universe of discourse (Ω) in to following seven subsets as

$$\Omega = \{x \in X/\mu_A(x) \le \mu_B(x) \le \mu_C(x)\} \cup \{x \in X/\mu_A(x) \le \mu_C(x) < \mu_B(x)\} \cup \{x \in X/\mu_B(x) < \mu_A(x) \le \mu_C(x)\} \cup \{x \in X/\mu_B(x) \le \mu_C(x) < \mu_A(x)\} \cup \{x \in X/\mu_C(x) < \mu_A(x) \le \mu_B(x)\} \cup \{x \in X/\mu_C(x) < \mu_B(x) < \mu_A(x)\}$$

which we will denote as $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$ respectively. We compute $F_{G,K}(A,B)$ in each of the subsets $\Omega_i, i = 1,2,3,4,5,6$. Then combine the results, thus we obtain $F_{G,K}(A,B)$ for the universe of discourse.

In Ω_1, $A \cup C \Leftrightarrow \mu_{A\cup C}(x) = \max\{\mu_A(x), \mu_C(x)\} = \mu_C(x)$

$$B \cup C \Leftrightarrow \mu_{B\cup C}(x) = \max\{\mu_B(x), \mu_C(x)\} = \mu_C(x)$$
$$A \cap C \Leftrightarrow \mu_{A\cap C}(x) = \min\{\mu_A(x), \mu_C(x)\} = \mu_A(x)$$
$$B \cap C \Leftrightarrow \mu_{B\cap C}(x) = \min\{\mu_B(x), \mu_C(x)\} = \mu_B(x)$$

Therefore, $F_{G,K}(A \cup C, B \cup C) = 0$ and $F_{G,K}(A \cap C, B \cap C) = F_{G,K}(A,B)$

Hence, $\max\{F_{G,K}(A \cup C, B \cup C), F_{G,K}(A \cap C, B \cap C)\} \le F_{G,K}(A,B)$ holds good for Ω_1, similarly the inequality holds for $\Omega_2, \Omega_3, \Omega_4, \Omega_5$ *and* Ω_6. Thus $F_{G,K}(A,B)$ is a valid measure of fuzzy directed divergence as represented graphically in Fig. 3 (where $a = p_i, b = q_i$ *and* $z = F_{G,K}^{i}(A,B)$).

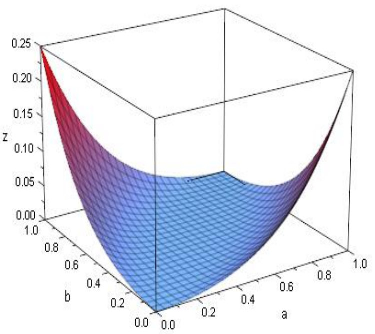

Figure 3. $F_{G,K}^{i}(A,B)$

Also, $$\frac{\partial F_{G,K}(A,B)}{\partial \mu_A(x_i)} = \left[\begin{array}{c}\frac{4(\mu_A(x_i)+\mu_B(x_i)-2)^3}{8(\mu_A(x_i)-1)^2+8(\mu_B(x_i)-1)^2}-2\mu_B(x_i)\\ +\frac{4(\mu_A(x_i)+\mu_B(x_i))^3}{8\mu_A(x_i)^2+8\mu_B(x_i)^2}-\frac{16\mu_A(x_i)(\mu_A(x_i)+\mu_B(x_i))^4}{(8\mu_A(x_i)^2+8\mu_B(x_i)^2)^2}\\ -\frac{(16\mu_A(x_i)-16)(\mu_A(x_i)+\mu_B(x_i)-2)^4}{(8(\mu_A(x_i)-1)^2+8(\mu_B(x_i)-1)^2)^2}+1\end{array}\right]$$

$$\frac{\partial F_{G,K}(A,B)}{\partial \mu_A(x_i)} = 0 \;\; when, \mu_A(x_i) = \mu_B(x_i)$$

Then,

$$\frac{\partial^2 F_{G,K}(A,B)}{\partial \mu_A(x_i)^2} = \left[\begin{array}{c}\frac{12(\mu_A(x_i)+\mu_B(x_i)-2)^2}{8(\mu_A(x_i)-1)^2+8(\mu_B(x_i)-1)^2}+\frac{16(\mu_A(x_i)+\mu_B(x_i)-2)^4}{(8(\mu_A(x_i)-1)^2+8(\mu_B(x_i)-1)^2)^2}\\ +\frac{12(\mu_A(x_i)+\mu_B(x_i))^2}{8\mu_A(x_i)^2+8\mu_B(x_i)^2}-\frac{16(\mu_A(x_i)+\mu_B(x_i))^4}{(8\mu_A(x_i)^2+8\mu_B(x_i)^2)^2}\\ +\frac{2(16\mu_A(x_i)-16)^2(\mu_A(x_i)+\mu_B(x_i)-2)^4}{(8(\mu_A(x_i)-1)^2+8(\mu_B(x_i)-1)^2)^3}-\frac{128\mu_A(x_i)(\mu_A(x_i)+\mu_B(x_i))^3}{(8\mu_A(x_i)^2+8\mu_B(x_i)^2)^2}\\ +\frac{512(\mu_A(x_i))^2(\mu_A(x_i)+\mu_B(x_i))^4}{(8\mu_A(x_i)^2+8\mu_B(x_i)^2)^3}-\frac{8(16\mu_A(x_i)-16)(\mu_A(x_i)+\mu_B(x_i)-2)^3}{(8(\mu_A(x_i)-1)^2+8(\mu_B(x_i)-1)^2)^2}\end{array}\right]$$

$$= \sum_{i=1}^{n} \frac{\{\partial^2 F_{G,K}(A,B)\}_i}{\partial \mu_A(x_i)^2}$$

Also $\frac{\partial^2 F_{G,K}(A,B)}{\partial \mu_A(x_i)^2} \ge 0$ for $0 \le \mu_A(x_i), \mu_B(x_i) \le 1$ *as* shown in Figure 4.

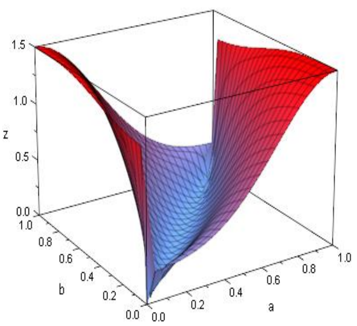

Figure 4. $\frac{\partial^2 F_{G,K}(A,B)}{\partial \mu_A(x_i)^2}$

Thus $F_{G,K}(A,B)$ has minimum value zero when $\mu_A(x_i) = \mu_B(x_i)$.

Thus $F_{G,K}(A,B)$ is a valid measure of fuzzy directed divergence. In next section, we have proved various properties related to proposed fuzzy directed divergence.

3. Properties of Proposed Fuzzy Directed Divergence

Measure $F_{G,K}(A,B)$, defined by (4) has the following properties:

Theorem 1: Let A and B two fuzzy sets then following properties can be verified for $F_{G,K}(A,B)$:

1) $F_{G,K}(A \cup B, A \cap B) = F_{G,K}(A,B)$
2) $F_{G,K}(A, A \cup B) = F_{G,K}(B, A \cap B)$

3) $F_{G,K}(A, A \cap B) = F_{G,K}(B, A \cup B)$
4) $F_{G,K}(A, \bar{A}) = n/4$, when A is a crisp set i.e. $\mu_A(x) = 0 \; or \; 1$

Proof: Divide the universe of discourse into two subsets as $\Omega = \{x \in X / \mu_A(x) \leq \mu_B(x)\} \cup \{x \in X / \mu_B(x) < \mu_A(x)\}$, which we will denote as Ω_1, Ω_2 respectively.

1) In $\Omega_1, A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$ and $A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) F_{G,K}(A \cup B, A \cap B) = F_{G,K}(B, A) = F_{G,K}(A, B)$.
Similarly we can prove the result for Ω_2. Hence 1) holds.
2) In Ω_1 , $A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$ and $A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) F_{G,K}(A, A \cup B) = F_{G,K}(A, B) = F_{G,K}(B, A) = F_{G,K}(B, A \cap B)$
In $\Omega_2, A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$ and $A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_B(x) F_{G,K}(A, A \cup B) = F_{G,K}(A, A) = 0 = F_{G,K}(B, B)$
$= F_{G,K}(B, A \cap B)$
Hence $F_{G,K}(A, A \cup B) = F_{G,K}(B, A \cap B)$

Similarly, 3) holds. Further, 4) holds due to membership values of crisp set.
Corollary 1: For any two fuzzy sets A and B, $F_{G,K}(A, A \cup B) + F_{G,K}(A, A \cap B) = F_{G,K}(A, B)$.
Proof: It follows from 2) and 3) of Theorem 1.
Corollary 2: For any two fuzzy sets A and B,
$F_{G,K}(B, A \cup B) + F_{G,K}(B, A \cap B) = F_{G,K}(B, A)$
Proof: It followed from 2) and 3) of Theorem1.

Theorem 2: For any fuzzy sets A, B and C, then
1) $F_{G,K}(A, B \cup C) + F_{G,K}(A, B \cap C) = F_{G,K}(A, B) + F_{G,K}(A, C)$
2) $F_{G,K}(A \cup B, C) + F_{G,K}(A \cap B, C) = F_{G,K}(A, C) + F_{G,K}(B, C)$

Proof: 1) Divide the universe of discourse into two subsets as $\Omega = \{x \in X / \mu_B(x) \geq \mu_C(x)\} \cup \{x \in X / \mu_B(x) < \mu_C(x)\}$, which we will denote as Ω_1, Ω_2 respectively. Then in Ω_1, $B \cup C \Leftrightarrow \mu_{B \cup C}(x) = \max\{\mu_B(x), \mu_C(x)\} = \mu_B(x)$. Then

$$F_{G,K}(A, B \cup C) = F_{G,K}(A, B) \tag{4}$$

And $B \cap C \Leftrightarrow \mu_{B \cap C}(x) = \min\{\mu_B(x), \mu_C(x)\} = \mu_C(x)$.Then

$$F_{G,K}(A, B \cap C) = F_{G,K}(A, C) \tag{5}$$

Adding equation 4 and 5, we obtain
$F_{G,K}(A, B \cup C) + F_{G,K}(A, B \cap C) = F_{G,K}(A, B) + F_{G,K}(A, C)$.
Similarly, we can also prove that the result hold in Ω_2. Analogously, 2) can also be proved.

Theorem 3: For any fuzzy set A, B and C,
1) $F_{G,K}(A \cup B, C) \leq F_{G,K}(A, C) + F_{G,K}(B, C)$
2) $F_{G,K}(A \cap B, C) \leq F_{G,K}(A, C) + F_{G,K}(B, C)$

Proof: 1) Divide the universe of discourse into two subsets as $\Omega = \{x \in X / \mu_A(x) \leq \mu_B(x)\} \cup \{x \in X / \mu_B(x) < \mu_A(x)\}$, which we will denote as Ω_1, Ω_2 respectively.

In $\Omega_1, A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$ and $A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$.
Let us consider the expression in Ω_1,
$F_{G,K}(A, C) + F_{G,K}(B, C) - F_{G,K}(A \cup B, C)$
$= F_{G,K}(A, C) + F_{G,K}(B, C) - F_{G,K}(B, C)$
$= F_{G,K}(A, C) \geq 0$
In $\Omega_2, A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$
and $A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$
$F_{G,K}(A, C) + F_{G,K}(B, C) - F_{G,K}(A \cup B, C)$
$= F_{G,K}(A, C) + F_{G,K}(B, C) - F_{G,K}(A, C)$
$= F_{G,K}(B, C) \geq 0$.
Similarly, 2) holds.

Theorem 4: For any two fuzzy sets A and B
1) $F_{G,K}(A, B) = F_{G,K}(\bar{A}, \bar{B})$
2) $F_{G,K}(A, \bar{B}) = F_{G,K}(\bar{A}, B)$

Proof: 1) Simply follows from the relation of membership value of an element and its complement.

2) Consider $F_{G,K}(A, \bar{B}) - F_{G,K}(\bar{A}, B)$

$$= \sum_{i=1}^{n} \left[\left[\frac{(\mu_A(x_i) + 1 - \mu_B(x_i))^4}{8(\mu_A(x_i)^2 + (1 - \mu_B(x_i))^2)} - \mu_A(x_i)(1 - \mu_B(x_i)) + \frac{(1 - \mu_A(x_i) + \mu_B(x_i))^4}{8((1 - \mu_A(x_i))^2 + (\mu_B(x_i))^2)} - (1 - \mu_A(x_i))(\mu_B(x_i)) \right] - \left[\frac{(1 - \mu_A(x_i) + \mu_B(x_i))^4}{8((1 - \mu_A(x_i))^2 + \mu_B(x_i)^2)} - (1 - \mu_A(x_i))\mu_B(x_i) + \frac{(1 + \mu_A(x_i) - \mu_B(x_i))^4}{8((\mu_A(x_i))^2 + (1 - \mu_B(x_i))^2)} - (\mu_A(x_i))(1 - \mu_B(x_i)) \right] \right]$$

$= 0$

This proves that
$F_{G,K}(A, \bar{B}) = F_{G,K}(\bar{A}, B)$.

4. Comparative Study

In this section, we compare the efficiency of proposed fuzzy directed divergence with existing fuzzy directed divergence measures given by Bhandari and Pal (1993), Fan and Xie (1999), Bhatia and Singh (2012) and Kapur (1997) at different values of α.

Let us consider two fuzzy sets $A = \{0.2, 0.9, 0.6, 0.1, 0.7\}$ and $B = \{0.6, 0.8, 0.1, 0.9, 0.5\}$. Calculated values of fuzzy divergence measures are given in Table 1.

Table 1. Comparative Study

	Fuzzy Directed Measure	Discrimination value
1	D(A,B)	1.286481
2	I(A,B)	0.650619
3	T(A,B)	0.215987
4	I_α(A,B) at α=0.1	0.276511
5	I_α(A,B) at α=0.2	0.544951
6	I_α(A,B) at α=0.3	0.810034
7	I_α(A,B) at α=0.9	2.587437
8	$F_{G,K}$ (A,B)	0.167164

As suggested in Tomar and Ohlan (2014) minimization of degree of difference depicts the efficiency of divergence measure. Table 1 clearly shown the value of the proposed divergence measure minimum as compared to other existing measure. It suggests that the measure of fuzzy divergence is more efficient than the existing divergence measures.

5. Application to Decision Problem

In this section, we present a method to solve decision-making problems using proposed fuzzy directed divergence.

Decision making is a process that involves many course of action under uncertain environment. In order to take decision the decision maker needs to select the best option from all available courses of action. To select the best course of action various research designed different divergence, similarity and entropy measures. Based on divergence measure, let us consider a decision-making problem involving a set of options $P = \{P_1, P_2, \dots\dots\dots, P_m\}$ to be considered on the basis of certain criteria $D = \{C_1, C_2, \dots\dots\dots, C_n\}$. For decision making, characteristic sets for each option are determined as assigning appropriate values to membership values and ideals solution P_* to the problem is having maximum membership values in each criterion. The divergence for each case is calculated and option with minimum divergence selected.

To exhibit the applicability of proposed fuzzy directed divergence, we consider few decision-making problems.

Example 1: Suppose customers want to buy a mobile connection. Customer wants to select a service provider from five options: A_1, A_2, A_3, A_4, A_5 mobile service providers on the basis of Network Quality (P_1), Triff Plan (P_2), Value Added Services (P_3) and Customer Care (P_4). For evaluating five alternatives, the decision makers formed five fuzzy sets as

$A_1 = \{(P_1, 0.5), (P_2, 0.6), (P_3, 0.3), (P_4, 0.2)\}$
$A_2 = \{(P_1, 0.7), (P_2, 0.7), (P_3, 0.7), (P_4, 0.4)\}$
$A_3 = \{(P_1, 0.6), (P_2, 0.5), (P_3, 0.5), (P_4, 0.6)\}$
$A_4 = \{(P_1, 0.8), (P_2, 0.6), (P_3, 0.3), (P_4, 0.2)\}$
$A_5 = \{(P_1, 0.6), (P_2, 0.4), (P_3, 0.7), (P_4, 0.5)\}$

Optimal solution is
$A_* = \{(P_1, 0.8), (P_2, 0.7), (P_3, 0.7), (P_4, 0.6)\}$

Divergence of A_*, w.r.t. each option given as
$D(A_1, A_*) = 0.027728$
$D(A_2, A_*) = 0.000876$
$D(A_3, A_*) = 0.002917$
$D(A_4, A_*) = 0.023099$
$D(A_5, A_*) = 0.005059$

Optimal solution is with minimum divergence is A_2 with preference order given as A_2 A_3, A_5, A_4 and A_1. So, customer should buy connection from operator A_2.

Example 2: A wants to open manufacturing plant and they need to select the location out of six locations $L_1, L_2, L_3, L_4, L_5, L_6$ on the basis of Business Climate (D_1), Infrastructure (D_2), Quality of Labor (D_3), Suppliers(D_4), Total Costs (D_5), Proximity to customers (D_6), Free Trade Zone (D_7). For evaluating six locations, the management formed six fuzzy sets as follows:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
L_1	0.4	0.7	0.5	0.9	0.4	0.6	0.6
L_2	0.7	0.9	0.6	0.7	0.6	0.6	0.8
L_3	0.9	0.6	0.4	0.5	0.7	0.5	0.3
L_4	0.5	0.5	0.6	0.3	0.6	0.8	0.7
L_5	0.6	0.5	0.7	0.6	0.7	0.5	0.5
L_6	0.4	0.3	0.2	0.5	0.5	0.4	0.3

Optimal solution is
$L_* = \{(D_1, 0.9), (D_2, 0.9), (D_3, 0.7), (D_4, 0.9), (D_5, 0.7), (D_6, 0.8), (D_7, 0.8)\}$

Divergence of L_* from each given option $L_1, L_2, L_3, L_4, L_5, L_6$ is given as
$D(L_1, L_*) = 0.038389$
$D(L_2, L_*) = 0.005637$
$D(L_3, L_*) = 0.056027$
$D(L_4, L_*) = 0.081290$
$D(L_5, L_*) = 0.038227$
$D(L_6, L_*) = 0.158639$

The optimal solution is with the minimum divergence. So, management should open manufacturing plant at location L_2 with preference order $L_2, L_5, L_1, L_3, L_4, L_6$.

6. Conclusions

In this paper we have presented a new measure of divergence based on aggregation operators and its properties are validated. The efficiency of the proposed fuzzy directed divergence measure has been presented by comparing it with some existing divergence measures. Further, the application of the proposed fuzzy divergence measure is discussed in decision making process. Finally, the introduced fuzzy directed divergence measure has been applied to a few illustrative examples of decision making problems, which shows how it helps in decision making by minimizing fuzzy directed divergence.

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