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Original Article

On the class of fuzzy number sequences bv_p^F

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Abstract

We introduce the notion of p-bounded variation of fuzzy real number sequences, bv_p^F, for $1 \le p < \infty$. We study some of its properties like completeness, monotonicity, convergence free, and symmetricity. Also, we prove some inclusion results.

Keywords: fuzzy real number, solid space, symmetric space, convergence free

1. Introduction

The concept of fuzzy set, a set whose boundary is not sharp or precise was introduced by L. A. Zadeh in 1965. It is the origin of the new theory of uncertainty, distinct from the notion of probability. After the introduction of fuzzy sets, the scope for studies in different branches of pure and applied mathematics increased widely. The notion of fuzzy sets was successfully applied in studying sequence spaces with classical metrics, for example by Altinok, Et and Çolak (2014), Altinok and Et (2015), Das (2014a, 2014b, 2017a, 2017b), Esi (2006, 2010a, 2010b, 2013), Et, Altinok, and Altin (2013), Nanda (1989), Tripathy and Dutta (2007), Tripathy and Das (2012), and Tripathy and Goswami (2015). Work with the concept of fuzzy metrics was done by Kelava and Seikkala (1984), Das (2014), Tripathy, Paul and Das (2015). In the field of fuzzy topology, some work was done by Dutta and Tripathy (2017), Tripathy and Debnath (2015), and Tripathy and Ray (2012).

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2. Definitions and Preliminaries

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on R, the real line. For $X, Y \in D$ we define

$$X \le Y, \text{ if } a_1 \le b_1 \text{ and } a_2 \le b_2,$$

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that (D, d) is a complete metric space. Also $\le$ is a partial order in D.

Definition 2.1 A fuzzy real number X is a fuzzy set on R, *i.e.* a mapping $X : R \to I\ (=[0,1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.2 A fuzzy real number X is called convex if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

Definition 2.3. If there exists a $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called normal.

Definition 2.4 A fuzzy real number X is said to be upper-semi continuous if, for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$. Throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

Definition 2.5. The α - *level* set $[X]^{\alpha}$ of the fuzzy real number X, for $0 < \alpha \le 1$, is defined by $[X]^{\alpha} = \{ t \in R: X(t) \ge \alpha \}$. If $\alpha = 0$, then it is the closure of the strong 0-cut. (The strong α-cut of the fuzzy real number X, for $0 \le \alpha \le 1$ is the set $\{t \in R : X(t) > \alpha \}$).

By *0-cut* or *0-level set* of the fuzzy real number X, we mean the closure of the strong 0-cut.

Throughout α means, $\alpha \in (0, 1]$ unless stated otherwise.

For $X, Y \in R(I)$ consider a partial ordering $\le$ as

$X \le Y$ if and only if $a_1^{\alpha} \le a_2^{\alpha}$ and $b_1^{\alpha} \le b_2^{\alpha}$, for all $\alpha \in (0,1]$, where $[X]^{\alpha} = \left[a_1^{\alpha}, b_1^{\alpha}\right]$ and $[Y]^{\alpha} = \left[a_2^{\alpha}, b_2^{\alpha}\right]$,

The arithmetic operations for α-level sets are defined as follows:

Let $X, Y \in R(I)$ and the α- level sets be $[X]^{\alpha} = \left[a_1^{\alpha}, b_1^{\alpha}\right]$, $[Y]^{\alpha} = \left[a_2^{\alpha}, b_2^{\alpha}\right]$, for $\alpha \in [0,1]$. Then

$$[X \oplus Y]^{\alpha} = \left[a_1^{\alpha} + a_2^{\alpha},\ b_1^{\alpha} + b_2^{\alpha}\right],$$

$$[X - Y]^{\alpha} = \left[a_1^{\alpha} - b_2^{\alpha},\ b_1^{\alpha} - a_2^{\alpha}\right],$$

$$[X \otimes Y]^{\alpha} = \left[\min_{i,j \in \{1,2\}} a_i^{\alpha} b_j^{\alpha},\ \max_{i,j \in \{1,2\}} a_i^{\alpha} b_j^{\alpha}\right],$$

$$\text{and } [Y^{-1}]^{\alpha} = \left[\frac{1}{b_2^{\alpha}}, \frac{1}{a_2^{\alpha}}\right],\ 0 \notin Y.$$

The set R of all real numbers can be embedded in $R(I)$. For $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \ne r. \end{cases}$$

For $r \in R$ and $X \in R(I)$, the scalar product rX is defined by

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \ne 0, \\ \bar{0}, & \text{if } r = 0. \end{cases}$$

Definition 2.6 The absolute value, $|X|$ of $X \in R(I)$ is defined by (Kaleva and Seikkala [1984])

$$|X|(t) = \begin{cases} \max(X(t), X(-t)), & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$

A fuzzy real number X is called non-negative if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$. The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \le \alpha \le 1} d\left([X]^{\alpha}, [Y]^{\alpha}\right).$$

Then $\bar{d}$ defines a metric on $R(I)$.

Definition 2.7 A fuzzy real numbers sequence (X_k) is said to be level convergent to the fuzzy real number X if, for each $\alpha \in [0, 1]$,

$$\lim_{k \to \infty} a_k^{\alpha} = a^{\alpha} \quad \text{and} \quad \lim_{k \to \infty} b_k^{\alpha} = b^{\alpha},$$

where $[X_k]^{\alpha} = [a_k^{\alpha}, b_k^{\alpha}]$, for all $k \in N$ and $[X]^{\alpha} = [a^{\alpha}, b^{\alpha}]$. If the convergence is uniform in α, then we say that (X_k) converges uniformly to X.

Definition 2.8 A sequence (X_k) of fuzzy real numbers is said to be convergent (uniformly) to the fuzzy real number X_0 if, for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \ge n_0$.

Definition 2.9 A fuzzy real number sequence (X_k) is said to be bounded if $|X_k| \le \mu$, for some $\mu \in R^*(I)$.

Definition 2.10 A class of sequences E^F is said to be normal (or *solid*) if $(Y_k) \in E^F$, whenever $\bar{d}(Y_k, \bar{0}) \le \bar{d}(X_k, \bar{0})$ for all $k \in N$ and $(X_k) \in E^F$.

Definition 2.11 Let $K = \{ k_1 < k_2 < k_3 \dots\} \subseteq N$ and E^F be a class of sequences. A *K-step set* of E^F is a set of sequences $\lambda_K^{E^F} = \{(X_{k_n}) \in w^F : (X_n) \in E^F\}$.

Definition 2.12 A canonical pre-image of a sequence $(X_{k_n}) \in \lambda_K^{E^F}$ is a sequence $(Y_n) \in w^F$, defined as follows:

$$Y_n = \begin{cases} X_n, & \text{if } n \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Definition 2.13 A canonical pre-image of a step set $\lambda_K^{E^F}$ is a set of canonical pre-images of all elements in $\lambda_K^{E^F}$, *i.e.*, Y is in canonical pre-image $\lambda_K^{E^F}$ if and only if Y is canonical pre-image of some $X \in \lambda_K^{E^F}$.

Definition 2.14 A class of sequences E^F is said to be *monotone* if E^F contains the canonical pre-images of all its step sets.

From the above definitions we have following remark.

Remark 2.15 A class of sequences E^F is solid $\Rightarrow$ E^F is monotone.

Definition 2.16 A class of sequences E^F is said to be symmetric if $(X_{\pi(n)}) \in E^F$, whenever $(X_k) \in E^F$, where π is a permutation of N.

Definition 2.17 A class of sequences E^F is said to be convergence free if $(Y_k) \in E^F$, whenever $(X_k) \in E^F$ *and* $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

Throughout the article w^F, ℓ_∞^F and c^F denote the class of *all*, *bounded* and *convergent* sequences of fuzzy real numbers respectively. The class of sequences ℓ_p^F, for $1 \le p < \infty$ of fuzzy real numbers was introduced and studied by Nanda (1989) as follows:

$$\ell_p^F = \left\{ X = (X_k) \in w^F : \sum_{k=1}^{\infty} \left\{ \bar{d}(X_k, \bar{0}) \right\}^p < \infty \right\}.$$

In this article we introduce the class of p-bounded variation sequences of fuzzy real numbers bv_p^F, for $1 \le p < \infty$ as follows:

$$bv_p^F = \left\{ X = (X_k) \in w^F : \sum_{k=1}^{\infty} \left\{ \bar{d}(\Delta X_k, \bar{0}) \right\}^p < \infty \right\},$$

where $\Delta X_k = X_k - X_{k+1}$, for all $k \in N$.

3. Main Results

Theorem 3.1 The class of sequences bv_p^F, $1 \le p < \infty$ is a complete metric space with the metric

$$\rho(X,Y) = \bar{d}(X_1, Y_1) + \left[\sum_{k=1}^{\infty} \left\{ \bar{d}(\Delta X_k, \Delta Y_k) \right\}^p \right]^{\frac{1}{p}},$$

where $X = (X_k)$, $Y = (Y_k) \in bv_p^F$.

Proof: Let $(X^{(n)})$ be a Cauchy sequence in bv_p^F, where $X^{(n)} = (X_k^{(n)}) = (X_1^{(n)}, X_2^{(n)}, X_3^{(n)}, \ldots) \in bv_p^F$, for all $n \in N$. Then, for each $0 < \varepsilon < 1$, there exists a positive integer n_0 such that for all $m, n \ge n_0$,

$$\rho(X^{(n)}, X^{(m)}) = \bar{d}(X_1^{(n)}, X_1^{(m)}) + \left[\sum_{k=1}^{\infty} \left\{ \bar{d}(\Delta X_k^{(n)}, \Delta X_k^{(m)}) \right\}^p \right]^{\frac{1}{p}} < \varepsilon.$$

It follows that $\bar{d}(X_1^{(n)}, X_1^{(m)}) < \varepsilon$, for all $m, n \ge n_0$. (1)

and $\left[\sum_{k=1}^{\infty} \left\{ \bar{d}(\Delta X_k^{(n)}, \Delta X_k^{(m)}) \right\}^p \right]^{\frac{1}{p}} < \varepsilon$, for all $m, n \ge n_0$. (2)

$\Rightarrow$ $\bar{d}(\Delta X_k^{(n)}, \Delta X_k^{(m)}) < \varepsilon$, for all $k \in N$ and $m, n \ge n_0$. (3)

Thus $(X_1^{(n)})$ and $(\Delta X_k^{(n)})$, for all $k \in N$ are Cauchy sequences in $R(I)$. Since $R(I)$ is complete, so $(X_1^{(n)})$ and $(\Delta X_k^{(n)})$, for all $k \in N$ are convergent in $R(I)$.

Let $\lim_{n\to\infty} X_1^{(n)} = X_1$ (4)

and $\lim_{n\to\infty} \Delta X_k^{(n)} = Z_k$, for all $k \in N$. (5)

From (4) and (5) we have,

$$\lim_{n\to\infty} X_k^{(n)} = X_k, \text{ for all } k \in N.$$

Now fix $n \ge n_0$ and let $m \to \infty$ in (1) and (2), we have

$\bar{d}(X_1^{(n)}, X_1) < \varepsilon$ and $\left[\sum_{k=1}^{\infty} \left\{ \bar{d}(\Delta X_k^{(n)}, \Delta X_k) \right\}^p \right]^{\frac{1}{p}} < \varepsilon$,

for all $n \ge n_0$ (6)

This implies that $\rho(X^{(n)}, X) < \varepsilon$, for all $n \ge n_0$. *i.e.* $X^{(n)} \to X$, as $n \to \infty$, where $X = (X_k)$.

Next, we show that $X \in bv_p^F$.

From (6) we have for all $n \ge n_0$,

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k^{(n)},\ \Delta X_k)\right\}^P < \varepsilon \cdot$$

Again for all $n\in N$, $X^{(n)} = (X_k^{(n)}) \in bv_p^F$

$$\Rightarrow \quad \sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k^{(n)},\bar{0})\right\}^P < \infty \cdot$$

Now for all $n \geq n_0$ we have,

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^P =$$

$$\left[\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\Delta X_k^{(n)})\right\}^P + \sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k^{(n)},\bar{0})\right\}^P\right] < \infty .$$

Hence $X \in bv_p^F$. This proves the completeness of bv_p^F .

Theorem 3.2 The class of sequences bv_p^F is neither monotone nor solid.

Proof: This result follows from the following example.

Example 3.1 Let us consider the sequence (X_k), defined as follows.

$$X_k(t) = \begin{cases} 1-3^{-1}k^{\frac{2}{p}}(t-2), & \text{for } 2 \leq t \leq 2+3k^{-\frac{2}{p}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $[\Delta X_k]^\alpha = \left[3(\alpha-1)(k+1)^{-\frac{2}{p}},\ 3(1-\alpha)k^{-\frac{2}{p}}\right]\cdot$

Thus $\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p = \sum_{k=1}^{\infty}\left\{3(1-\alpha)k^{-\frac{2}{p}}\right\}^p < \infty.$

Therefore, $(X_k)\in bv_p^F$.

Let $J = \{k\in N : k = 2i-1,\ i\in N\}$ be a subset of N and let $\overline{\overline{(bv_p^F)}}_J$ be the canonical pre-image of the J-step set $(bv_p^F)_J$ of bv_p^F, defined as follows:

$(Y_k)\in \overline{\overline{(bv_p^F)}}_J$, is the canonical pre-image of $(X_k)\in bv_p^F$ implies

$$Y_k = \begin{cases} X_k, & \text{for } k \in J, \\ \bar{0}, & \text{for } k \notin J. \end{cases}$$

Then

$$[Y_k]^\alpha = \begin{cases} \left[2,\ \left\{2+3(1-\alpha)k^{-\frac{2}{p}}\right\}\right], & \text{for } k \in J, \\ [0,0], & \text{for } k \notin J \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} \left[2,\ \left\{2+3(1-\alpha)k^{-\frac{2}{p}}\right\}\right], & \text{for } k \in J, \\ \left[-\left\{2+3(1-\alpha)(k+1)^{-\frac{2}{p}}\right\},\ -2\right], & \text{for } k \notin J. \end{cases}$$

Therefore,

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta Y_k,\bar{0})\right\}^P =$$

$$\sum_{k\in J}\left\{2+3(1-\alpha)k^{-\frac{2}{p}}\right\}^p + \sum_{k\notin J}\left\{2+3(1-\alpha)(k+1)^{-\frac{2}{p}}\right\}^p$$

$$\geq 2^p \sum_{k\in J}\left\{2^p + 3(1-\alpha)^p k^{-2}\right\}$$,which is unbounded.

Thus $(Y_k)\notin bv_p^F$. Hence bv_p^F is not monotone.

The class bv_p^F is not solid which follows from the remark 2.15.

Theorem 3.3 The class of sequences bv_p^F is not convergence free.

Proof: The result follows from the following example.

Example 3.2 Consider the sequence $(X_k)\in bv_p^F$ defined as follows:

For k even,

$$X_k(t) = \begin{cases} 1+k^{\frac{2}{p}}t, & \text{for } -k^{-\frac{2}{p}} \leq t \leq 0, \\ 1-k^{\frac{2}{p}}t, & \text{for } 0 < t \leq k^{-\frac{2}{p}}, \\ 0, & \text{otherwise} \end{cases}$$

and for k odd, $X_k = \bar{0}$.

Then

$$[X_k]^\alpha = \begin{cases} \left[(\alpha-1)k^{-\frac{2}{p}},(1-\alpha)k^{-\frac{2}{p}}\right], & \text{for } k \text{ even}, \\ [0,0], & \text{for } k \text{ odd} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[\left(\alpha-1\right)(k+1)^{-\frac{2}{p}},\ \left(1-\alpha\right)(k+1)^{-\frac{2}{p}}\right], & \text{for } k \text{ odd,} \\ \left[\left(\alpha-1\right)k^{-\frac{2}{p}},\ \left(1-\alpha\right)k^{-\frac{2}{p}}\right], & \text{for } k \text{ even.} \end{cases}$$

Therefore, $\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^P = 2\sum_{i=1}^{\infty}\left\{\frac{1-\alpha}{\left(2i\right)^{\frac{2}{p}}}\right\}^p < \infty.$

Thus $(X_k)\in bv_p^F$.

Let us define a sequence (Y_k) as follows:
For k odd, $Y_k = \bar{0}$
and for k even,

$$Y_k(t) = \begin{cases} 1+k^{\frac{1}{p}}t, & \text{for } -k^{-\frac{1}{p}} \le t \le 0, \\ 1-k^{\frac{1}{p}}t, & \text{for } 0 < t \le k^{-\frac{1}{p}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$[Y_k]^\alpha = \begin{cases} [0,0], & \text{for } k \text{ odd,} \\ \left[\left(\alpha-1\right)k^{-\frac{1}{p}},\ \left(1-\alpha\right)k^{-\frac{1}{p}}\right], & \text{for } k \text{ even.} \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} \left[\left(\alpha-1\right)k^{-\frac{1}{p}},\ \left(1-\alpha\right)k^{-\frac{1}{p}}\right], & \text{for } k \text{ even,} \\ \left[\left(\alpha-1\right)(k+1)^{-\frac{1}{p}},\ \left(1-\alpha\right)(k+1)^{-\frac{1}{p}}\right], & \text{for } k \text{ odd.} \end{cases}$$

Thus

$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta Y_k,\bar{0})\right\}^P = 2\sum_{i=1}^{\infty}\left\{\frac{1-\alpha}{\left(2i\right)^{\frac{1}{p}}}\right\}^p$, which is unbounded.

i.e., $(Y_k)\notin bv_p^F$. Hence bv_p^F, is not convergence free.

Theorem 3.4 The class of sequences bv_p^F, $p>1$ is not symmetric.

Proof: The result follows from the following example.

Example 3.3 Consider a sequence $(X_k)\in bv_p^F$ defined as follows:

$$X_1(t) = \begin{cases} 1, & \text{for } -\frac{1}{2} \le t \le 0, \\ 0, & \text{otherwise} \end{cases}$$

and for $k \ge 2$,

$$X_k(t) = \begin{cases} 1, & \text{for } -\left\{\sum_{r=1}^{k-1}\left(\frac{1}{r}\right)+\frac{1}{2k}\right\} \le t \le -\sum_{r=1}^{k-1}\left(\frac{1}{r}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\left[X_1\right]^\alpha = \left[-\frac{1}{2},\ 0\right]$

and for $k \ge 2$,

$$[X_k]^\alpha = \left[-\left\{\sum_{r=1}^{k-1}\left(\frac{1}{r}\right)+\frac{1}{2k}\right\},\ -\sum_{r=1}^{k-1}\left(\frac{1}{r}\right)\right].$$

Now for all $k\in N$, $[\Delta X_k]^\alpha$ =

$$\left[-\left\{\frac{1}{2k}-\frac{1}{k}\right\}, \left\{\frac{1}{k}+\frac{1}{2(k+1)}\right\}\right] = \left[\frac{1}{2k}, \left\{\frac{1}{k}+\frac{1}{2(k+1)}\right\}\right].$$

For $p > 1$ we have,

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^P = \sum_{k=1}^{\infty}\left\{\frac{1}{k}+\frac{1}{2(k+1)}\right\}^p \le$$

$$2^p\sum_{k=1}^{\infty}\left\{\frac{1}{k^p}+\frac{1}{2^p(k+1)^P}\right\} < \infty.$$

Thus, $(X_k)\in bv_p^F$, $p>1$.

Let (Y_k) be a rearrangement of the sequence (X_k), defined by

$(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, X_7, X_{36}, X_8, X_{49}, X_{10}, \ldots)$.

i.e. $Y_k = X_{\left(\frac{k+1}{2}\right)^2}$, for k odd,

$= X_{\left(n+\frac{k}{2}\right)}$, for k even and $n\in N$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$.

Then for $k = 1$, we have
$\left[\Delta Y_k\right]^\alpha = \left[\Delta Y_1\right]^\alpha = \left[X_1\right]^\alpha - \left[X_2\right]^\alpha$ = [0.5, 1.25].

Again,

for k odd with $k>1$ and $n\in N$, satisfying $n(n-1) < \frac{k+1}{2} \le n(n+1)$,

$$[\Delta Y_k]^\alpha = \left[X_{\left(\frac{k+1}{2}\right)^2}\right]^\alpha - \left[X_{\left(n+\frac{k+1}{2}\right)}\right]^\alpha$$

$$= \left[-\left\{\sum_{r=\left(n+\frac{k+1}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{r}+\frac{1}{2\left(\frac{k+1}{2}\right)^2}\right\},\ -\left\{\sum_{r=\left(n+\frac{k+1}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{r}\right\}+\frac{1}{2\left(n+\frac{k+1}{2}\right)}\right]$$

and

for k even and $n \in N$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$,

$$[\Delta Y_k]^\alpha = \left[X_{\left(n+\frac{k}{2}\right)}\right]^\alpha - \left[X_{\left(\frac{k+2}{2}\right)^2}\right]^\alpha$$

$$= \left[\left\{\sum_{r=\left(n+\frac{k}{2}\right)}^{\left(\frac{k+2}{2}\right)^2-1}\frac{1}{r}\right\}-\frac{1}{2\left(n+\frac{k}{2}\right)},\ \left\{\sum_{r=\left(n+\frac{k}{2}\right)}^{\left(\frac{k+2}{2}\right)^2-1}\frac{1}{r}+\frac{1}{2\left(\frac{k+2}{2}\right)^2}\right\}\right]$$

It is observed that the distance of $[\Delta Y_k]^\alpha$ from $[\bar{0}]^\alpha$, for all (odd and even) $k\in N$ is numerically greater than 0.1. Therefore, $\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta Y_k,\bar{0})\right\}^P$ is unbounded for $p > 1$.

Thus $(Y_k)\notin bv_p^F$, $p > 1$. Hence bv_p^F, $p > 1$ is not symmetric.

Theorem 3.5

(*a*) $\ell_p^F \subset bv_p^F$, *for* $1 < p < \infty$ and the inclusion is strict.

(*b*) $bv_q^F \subset bv_p^F$, *for* $1 \le q < p < \infty$ and the inclusion is strict.

(*c*) $bv^F \subset bv_p^F$, *for* $1 < p < \infty$ and the inclusion is strict.

Proof: (*a*) Let us consider a sequence $(X_k)\in \ell_p^F$.

Then $\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^P < \infty$.

Again, $\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^P$

$$\le 2^p\left[\sum_{k=1}^{\infty}\left\{\left(\bar{d}(X_k,\bar{0})\right)^p+\left(\bar{d}(X_{k+1},\bar{0})\right)^p\right\}\right]<\infty.$$

Therefore, $\ell_p^F \subset bv_p^F$.

The strictness of the inclusion follows from the following example.

Example 3.4 Consider a sequence (X_k) defined by

$$X_k(t) = \begin{cases} 1+k^{\frac{2}{p}}(t-2), & \text{for } 2-k^{-\frac{2}{p}} \le t \le 2, \\ 1-2^{-1}k^{\frac{2}{p}}(t-2), & \text{for } 2 < t \le 2+2k^{-\frac{2}{p}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \left[\left\{2+(\alpha-1)k^{-\frac{2}{p}}\right\},\ \left\{2+2(1-\alpha)k^{-\frac{2}{p}}\right\}\right]$$

and

$$[\Delta X_k]^\alpha = \left[-\left\{(1-\alpha)k^{-\frac{2}{p}}+2(1-\alpha)(k+1)^{-\frac{2}{p}}\right\},\right.$$

$$\left.\left\{2(1-\alpha)k^{-\frac{2}{p}}+(1-\alpha)(k+1)^{-\frac{2}{p}}\right\}\right].$$

Thus

$$\sum_{k=1}^{\infty}\left\{\bar{d}(X_k,\bar{0})\right\}^P = \sum_{k=1}^{\infty}\left\{2+2(1-\alpha)k^{-\frac{2}{p}}\right\}^p,$$

which is unbounded. Hence $(X_k) \notin \ell_p^F$. Next,

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^P = \sum_{k=1}^{\infty}\left\{2(1-\alpha)k^{-\frac{2}{p}}+(1-\alpha)(k+1)^{-\frac{2}{p}}\right\}^p$$

$$\le 2^p(1-\alpha)^p\left[\sum_{k=1}^{\infty}\left\{2^p k^{-2}+(k+1)^{-2}\right\}\right]$$

$$< \infty.$$

i.e. $(X_k)\in bv_p^F$. Therefore the inclusion is strict.

(*b*) Let $(X_k)\in bv_q^F$. Then $\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^q < \infty$.

Since $\Delta X_k \to \bar{0}$, as $k \to \infty$, so there exists an positive integer n_0 such that

$$\bar{d}(\Delta X_k,\bar{0}) \le 1, \text{ for all } k > n_0.$$

We have

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p = \sum_{k=1}^{n_0-1}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p + \sum_{k=n_0}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p. \quad (7)$$

Clearly,

$$\sum_{k=n_0}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p \le \sum_{k=n_0}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^q < \infty, \text{ for } p > q$$

and $\sum_{k=1}^{n_0-1}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p$ is a finite sum.

Hence (7) $\Rightarrow \sum_{k=n_0}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p < \infty \Rightarrow (X_k) \in bv_p^F$ and thus $bv_q^F \subset bv_p^F$.

The strictness of the inclusion follows from the following example.

Example 3.5 Consider the sequence (X_k) such that $\Delta X_k = k^{-\frac{1}{q}}$, for all $k \in N$.

Then

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^q = 1+\left(2^{-\frac{1}{q}}\right)^q+\left(3^{-\frac{1}{q}}\right)^q+\dots = \sum_{k=1}^{\infty}\frac{1}{k},$$

which is unbounded.

Hence $(X_k) \notin bv_q^F$.

But,

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p = \left(1^{-\frac{1}{q}}\right)^p+\left(2^{-\frac{1}{q}}\right)^p+\left(3^{-\frac{1}{q}}\right)^p+\dots$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^r} < \infty, \text{ where } r = \frac{p}{q} > 1.$$

Hence $(X_k) \in bv_p^F$. Thus the inclusion is strict.

(c) The proof is obvious, so omitted. The strictness of the inclusion follows from the following example.

Example 3.6 Consider the sequence (X_k) defined in example 3.3.

Then

$$\sum_{k=1}^{\infty}\bar{d}(\Delta X_k,\bar{0}) = \sum_{k=1}^{\infty}\left\{\frac{1}{k}+\frac{1}{2(k+1)}\right\}, \text{ which is unbounded.}$$

Hence $(X_k) \notin bv^F$.

But,

$$\sum_{k=1}^{\infty}\left\{\bar{d}(\Delta X_k,\bar{0})\right\}^p = \sum_{k=1}^{\infty}\left\{\frac{1}{k}+\frac{1}{2(k+1)}\right\}^p$$

$$\le 2^p\sum_{k=1}^{\infty}\left\{\frac{1}{k^p}+\frac{1}{2^p(k+1)^p}\right\} < \infty$$

$\Rightarrow (X_k) \in bv_p^F$.

Hence the inclusion is proper.

4. Conclusions

We introduced the notion of p-bounded variation of fuzzy real number sequences, bv_p^F, for $1 \le p < \infty$ and studied some of its properties like completeness, monotonicity, convergence free, and symmetricity. We also established some inclusion results involving this space. The methodology adopted to establish the results can be applied to study the class of double bounded variation sequences. Also the space introduced can be studied from a fuzzy metric point of view.

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