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Contributed Paper

Numerical Simulation for Multi-asset Derivatives Pricing Under Black-Scholes Model

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ABSTRACT

Options are one of the most important financial instruments for risk management and option pricing is a well known problem in financial mathematics. In this paper, we simulate the multi-asset option price using the Monte Carlo method with variance reduction techniques. Based on this simulation study, we found that the Mean Monte Carlo method performs better than the Crude Monte Carlo method.

Keywords: multi-asset options, European option pricing, Monte Carlo, variance reduction, Black-Scholes model
Mathematics Subject Classification (2000): 65C05, 91B28

1. INTRODUCTION

Any economic activity is associated with risk. An option is one of the most important financial instruments for managing risk [7]. Options are financial derivative products that give the right, but not the obligation, to engage in a future transaction on some underlying financial instrument. For this propose, option pricing is very important and constantly expanding. Due to the variety of options, nowadays many models and methods introduced for option pricing [1-6]. Unfortunately, for pricing of many options there exists no closed-form. This limitation specifically on multi-asset options is more tangible than one-asset options. Among the most powerful and broadly applicable numerical methods available for valuing financial instruments are the Monte Carlo simulation methods. A Monte Carlo calculation seeks an estimator $\bar{X}$ to some expected value $E[X]$ [2]. The goal of variance reduction methods is to achieve a more precise estimate of some expected value than could be obtained in a purely analog calculation using the same computational effort. For "successes over trials" problems in which a success is very unlikely, variance reduction tries to bias the calculation so that more successes are obtained. For problems in which a continuum of outcomes is possible, variance reduction strives to reduce the spread among the history results and bring them closer to the mean [6]. To increase the accuracy of Monte Carlo estimate there are a variety of variance reduction techniques such as control variates, antithetic variates, and conditional Monte Carlo procedures [10, 12, 15]. In this paper, we compare the performance of these methods to estimate the price of the European basket options. The rest of the paper is as follows: European options and European basket options introduce in section 2. In Section 3, we describe Monte Carlo estimator and variance reduction techniques. Finally, we present simulation results in section 4.

2. EUROPEAN OPTIONS

A call option (put option) gives the right, but not the obligation, to buy (sell) an underlying asset at a fixed price (exercise price or strike price) at or before a specified date (maturity date or expiry date). Gain or loss on the option is called payoff. The simplest option is the European option that can be exercised only on the maturity. Let $\boldsymbol{S(T)}$ denote the underlying asset price at maturity and **K** be the exercise price. For a call option, if $\boldsymbol{S(T) > K}$ then the holder of the option exercise it for a profit of $\boldsymbol{S(T) - K}$. On the other hand, if $\boldsymbol{S(T) \leq K}$ the option expires worthless. Thus the payoff function of the European call option at the maturity date is as follows

$$(S(T)-K)^{+} = \max\{S(T)-K,0\}.$$

Based on the risk-neutral valuation method [4] the price of the call option is equal to expectation of discounted payoff with risk-free interest rate, i.e.

$$C = e^{-rT} E[\max\{S(T)-K,0\}]. \tag{1}$$

One of the most important models for describing underlying asset price is the Black-Scholes model [1]. Based on this model, the price of the underlying asset is assumed to follow the Geometric Brownian Motion and thus satisfy in the following stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \qquad \mu \in \quad , \ \sigma > 0 \tag{2}$$

where B(t) is a Brownian motion. The parameter μ, σ is called drift and volatility, respectively. By using Ito's lemma, the unique solution of the above stochastic differential equation at maturity T is given by [9]

$$S(T) = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B(T)\right\} \tag{3}$$

here S_0 is the initial price of the underlying asset, assumed to be known. If Z be a standard normal random variable, the random variables $\sqrt{T}Z$ and $dB(T)$ have identical distributions. Therefore, Eq. (3) can be rewritten as follows

$$S(T) = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right\} \tag{4}$$

According to the risk neutral valuation principle, drift parameter μ is equal to the risk-free interest rate r. Using the relationship between normal and log-normal distributions and based on the above equation, it can be shown that $S(T)$ has a log-normal distribution with mean $(\mu - \frac{1}{2}\sigma^2)T + \ln S_0$ and variance $\sigma^2 T$. Therefore, we have

$$S(T) = S_0 e^{\mu T} \tag{5}$$

We can easily seen that if Z be a standard normal random variable and a, b and c be positive constants, then we will have

$$E[\max\{ae^{bZ} - c, 0\}] = a e^{b^2/2}\Phi(b + \frac{1}{b}\log\frac{a}{c}) - c\,\Phi(\frac{1}{b}\log\frac{a}{c})$$

where the function $\Phi(.)$ is the cumulative distribution function of the standard normal distribution. Thus, by substituting $a = S_0 e^{-\frac{\sigma^2 T}{2}}$, $b = \sigma\sqrt{T}$ and $c = K$ in the above equation and using Eq. (4) we conclude

$$\begin{aligned} C = e^{-rT} E[\max\{S(T) - K, 0\}] = e^{-rT} e^{-rT} S_0 \Phi\left(\frac{1}{\sigma\sqrt{T}}\log\frac{S_0 e^{rT}}{K} + \frac{\sigma\sqrt{T}}{2}\right) \\ - Ke^{-rT}\Phi\left(\frac{1}{\sigma\sqrt{T}}\log\frac{S_0 e^{rT}}{K} - \frac{\sigma\sqrt{T}}{2}\right) \\ = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \end{aligned} \tag{6}$$

The values d_1 and d_2 are given by

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \tag{7}$$

2.1 European Basket options

Options issued on multiple underlying assets called multi-asset options. One of the most important of multi-asset options is basket option. A European basket option holder, has the right to buy or sell a collection of underlying assets. The Payoff function of European call basket option is as follows

$$(\sum_{i=1}^{n} w_i S_i(T) - K)^+ = \max\{\sum_{i=1}^{n} w_i S_i(T) - K, 0\} \tag{8}$$

where n is the number of underlying assets, W_i is the weight of the i^{th} underlying asset and is its price at maturity T. In the Black-Scholes model framework, every underlying asset satisfies the stochastic differential Eq. (2), i.e.

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i dB_i(t) \quad i = 1,\ldots,n. \tag{9}$$

in which $B_i(t)$, $i = 1, 2, \ldots, n$ are correlated Brownian motions with correlation ρ_{ij} $(i \neq j)$. Let Σ be the covariance matrix of these Brownian motion. By applying Cholesky decomposition for Σ (i.e. $\Sigma = VV^t$ and V is a unique lower triangular matrix given that Σ being symmetric and positive definite) and based on the independent Brownian motions $W_i(t)$, Eq. (9) can be written as

$$dS_i(t) = \mu_i S_i(t)dt + \sum_{i=1}^{n} v_{ij} dW_i(t) \quad i = 1, \ldots, n \tag{10}$$

where v_{ij} is the ij^{th} element of the matrix V. This decomposition is used to simulate correlated prices of underlying assets [8]. Unlike the one-asset European option whose price evaluates from Eq. (6) there is not a closed-form for multi-asset option price. In [11] derived a closed-form for these options by assuming that the underlying asset price has a reciprocal gamma distribution.

3. Variance Reduction Techniques for Monte Carlo Estimator

Monte Carlo method is an efficient technique to evaluate

$$\theta = E[g(X)] = \int g(x) f(x) dx$$

where $f(x)$ is the probability function of the random variable X and $g(.)$ is any arbitrary function. For this propose, random numbers $X_1, \ldots, X_n$ were generate from $f(x)$ and average of $g(x_i)$'s is calculated as Monte Carlo estimator, $\hat{\theta} = \sum_{i=1}^{N} g(x_i) \Big/ N$. This estimator is called the crude Monte Carlo estimator. By assuming independent of X_i's, the variance of $\hat{\theta}$ is equal to σ^2/N where σ^2 is the variance of the random variable X. There are many techniques for reducing the variance of Monte Carlo estimator [12,15]. In this paper, we focus on the deterministic version of them, namely, antithetic variates, control variates and a new control variate introduced by Pellizzari [12] which is called Mean Monte Carlo.

3.1 Antithetic Variates Technique

Consider the parameter $\theta = E[g(X)]$. Let Y_1 and Y_2 are two Monte Carlo estimators for parameter θ. A new estimator can be defined by $\theta = (Y_1 + Y_2)/2$. The variance of this estimator is as follows

$$\text{Var}(\hat{\theta}) = \frac{1}{4}\{\text{Var}(Y_1) + \text{Var}(Y_2) + 2\text{Cov}(Y_1, Y_2)\}.$$

If $\text{Cov}(Y_1, Y_2) < 0$ then the new estimator $\hat{\theta}$ has smaller variance than Y_1, Y_2 [6]. It is well known that if U be a uniform random variable on [0,1] then $1-U$ also will be uniformly distributed on this interval. As a result $g(U)$ and $g(1-U)$ will be unbiased estimators for θ. It is shown that [3] if $g(U)$ be a non-decreasing or non-increasing function of U, then $\text{Cov}(g(U), g(1-U)) < 0$. Thus, for random numbers $U_1, \ldots, U_N$ from [0,1] two Monte Carlo estimators $\hat{\theta}_1 = \sum_{i=1}^{N} g(u_i) \Big/ N$ and $\hat{\theta}_2 = \sum_{i=1}^{N} g(1-u_i) \Big/ N$ can be combined to get the antithetic Monte Carlo estimator as follows

$$\hat{\theta}_{AV} = \frac{\hat{\theta}_1 + \hat{\theta}_2}{2} = \frac{1}{2N}\sum_{i=1}^{N}\left[g(u_i) + g(1-u_i)\right] \tag{11}$$

where U and $1-U$ is called antithetic variates.

3.2 Control Variates Technique

Consider the parameter $\theta = E[Y]$ and suppose X be a random variable with known expectation. We know that X and Y are unbiased estimators for their expectations. With these estimators a general class of unbiased estimators for θ can be constructed as follows

$$\hat{\theta}_{CV} = Y + c(X - E[X]) \tag{12}$$

where c is a real number. We want to choose c such that to minimize Var($\hat{\theta}_{CV}$) that has the form

$$\mathrm{Var}(\hat{\theta}_{CV}) = \mathrm{Var}(Y) + c^2\mathrm{Var}(X) + 2c\,\mathrm{Cov}(X, Y) \tag{13}$$

with differentiating from Eq. (13) and equal to zero we obtain the optimal c

$$c^* = -\frac{\mathrm{Cov(Y,Y)}}{\mathrm{Var(X)}} \tag{14}$$

By substituting c^* in Eq. (13), we obtain

$$\begin{aligned} \mathrm{Var}(\hat{\theta}_{CV}) &= \mathrm{Var}(Y) - \frac{\mathrm{Cov}(Y,X)}{\mathrm{Var}(X)} \\ &= \mathrm{Var}(\hat{\theta}) - \frac{\mathrm{Cov}^2(Y,X)}{\mathrm{Var}(X)} \end{aligned} \tag{15}$$

Thus to achieve variance reduction we must have Cov(Y, X)≠0.

In the above process, the random variable X is called control variates for random variable Y. To be more precise, is a control variate for Y if it is correlated with Y and its expectation is known. Therefore, the control variates Monte Carlo estimator is defined as follows

$$\hat{\theta}_{CV} = \overline{Y} + c^*(\overline{X} - E(X)) \tag{16}$$

in which $\overline{Y}$ and $\overline{X}$ are calculate based on N random sample from (Y, X). To use the above estimator we have to estimate Cov(Y, X). It can be shown that $c^* = -\hat{b}$ such that $\hat{b}$ is the least squares estimator for the slope in the below regression equation [14]

$$Y = a + bX + e, \quad e \sim N(0, \sigma^2) \tag{17}$$

Thus control variate Monte Carlo estimator can be rewrite as follows

$$\hat{\theta}_{CV} = \overline{Y} + c^*(\overline{X} - E(X)) = \overline{Y} - \hat{b}(\overline{X} - E(X)) = \hat{a} + \hat{b}E[X] \tag{18}$$

where $\hat{a} = \overline{Y} - \hat{b}\overline{X}$ is the least squares estimator of the intercept in Eq. (17).

Monte Carlo estimator with one control variate can be extended to more than one control variate as follows

$$\hat{\theta}_{CV} = \overline{Y} + c_1^*(\overline{X}_1 - E[X_1]) + \ldots + c_m^*(\overline{X}_m - E[X_m]) \tag{19}$$

in which X_i's are control variants and it is assumed their expectations is known.

In the simulations for option pricing, the price of the underlying asset at maturity date is used as a control variate. It is obvious that in the European basket option there exist more than one control variate in the form $S_{1T},...,S_{nT}$. Moreover, in this paper we use a new control variate introduced by Pellizzari which is called Mean Monte Carlo [12]. In this method, i^{th} control variate is as follows

$$g_i(T) = \max\left\{ \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j E[S_j(T)] + w_i S_i(T) - K, 0 \right\}, \quad i = 1,...,n$$

here $E[S_j(T)]$ is the expectation of j^{th} underlying asset can be obtained by Eq. (5). Thus, $g_i(T)$ is equal to [14]

$$g_i(T) = w_i \max\left\{ S_i(T) - \tilde{K}_i, 0 \right\}, \quad i = 1,...,n \tag{20}$$

where

$$\tilde{K}_i = \frac{1}{w_i}\left(K + w_i E[S_i(T)] - \sum_{j=1}^{n} w_j E[S_j(T)] \right).$$

By the above equation, the expectation of $g_i(T)$ can be calculated by Black-Scholes formula in Eq. (6).

4. SIMULATION RESULTS

Example 1. In this section, we use Monte Carlo simulation to price a European call basket option. We also compare the performance of variance reduction techniques to reduce the variance of Monte Carlo estimator. Simulations were done for the following values

$$S_{0i} = 100, \quad \sigma_i^2 = 0.2, \quad r = 0.1, \quad K = 100,$$
$$T = 1, \; w_i = \frac{1}{n}, \quad \rho_{ij} = 0.5, \quad (i \neq j)\; i, j = 1,...,n.$$

where n is the number of underlying assets. For $w_i = \frac{1}{n}$, $i = 1,\ldots,n$ the payoff of the basket option will depend on the arithmetic average of prices at maturity T. The simulation results[1] for $n = 4$ are shown in Table 1. In Tables 1 and 2, N denotes the number of replications, $\hat{C}$ is the Monte Carlo estimate of the option price and SD is the corresponding standard deviation of the estimators. Column named by crude MC belongs to Monte Carlo estimator without variance reduction techniques. Also, AV is referred to Antithetic Variates technique, CV and MMC are related to final price and Pellizzari method (Mean Monte Carlo method) and Control Variates technique.

[1] The Monte Carlo simulation is done in R software.

Table 1. Simulation results for European basket option for n = 4.

N	Crude MC		AV		CV		MMC	
	$\hat{C}$	SD	$\hat{C}$	SD	$\hat{C}$	SD	$\hat{C}$	SD
1000	11.74	0.42	12.02	0.16	11.93	0.12	11.93	0.12
2000	11.71	0.28	11.72	0.11	11.75	0.08	11.76	0.08
3000	12.26	0.24	11.87	0.09	11.84	0.07	11.84	0.06
4000	11.96	0.21	11.92	0.08	11.94	0.06	11.94	0.06
5000	11.98	0.18	11.89	0.07	11.90	0.05	11.90	0.05
6000	11.78	0.17	11.85	0.06	11.87	0.05	11.89	0.05
7000	11.90	0.16	11.91	0.06	11.91	0.04	11.92	0.04
8000	12.10	0.15	11.97	0.06	11.94	0.04	11.94	0.04
9000	11.90	0.14	11.82	0.05	11.84	0.04	11.84	0.04
10000	12.03	0.13	11.88	0.05	11.90	0.04	11.90	0.04

Table 2: Simulation results for European basket option for n = 10.

N	Crude MC		AV		CV		MMC	
	$\hat{C}$	SD	$\hat{C}$	SD	$\hat{C}$	SD	$\hat{C}$	SD
1000	11.59	0.40	11.65	0.14	11.58	0.11	11.58	0.11
2000	11.57	0.28	11.69	0.10	11.64	0.08	11.64	0.08
3000	11.84	0.23	11.55	0.08	11.56	0.06	11.56	0.06
4000	11.52	0.19	11.52	0.07	11.52	0.05	11.52	0.05
5000	11.64	0.18	11.67	0.07	11.64	0.05	11.64	0.05
6000	11.71	0.16	11.64	0.06	11.63	0.04	11.63	0.04
7000	11.68	0.15	11.68	0.05	11.65	0.04	11.65	0.04
8000	11.49	0.14	11.68	0.05	11.66	0.04	11.66	0.04
9000	11.74	0.13	11.61	0.05	11.60	0.04	11.60	0.04
10000	11.71	0.12	11.58	0.04	11.60	0.03	11.60	0.03

Based on the results in these tables, the standard deviation of variance reduction techniques, for every N, are always smaller than the standard deviation of Crude Monte Carlo. Among these methods, the performance of CV and MMC is better than AV method. On the other hand, CV and MMC methods are very similar to each other. Obtained results are shown in Figures 1 and 2. Simulation results for European basket option for $n = 10$ are shown in Table 2. The numerical results shown that, the performance of CV and MMC methods are completely similar to each other.

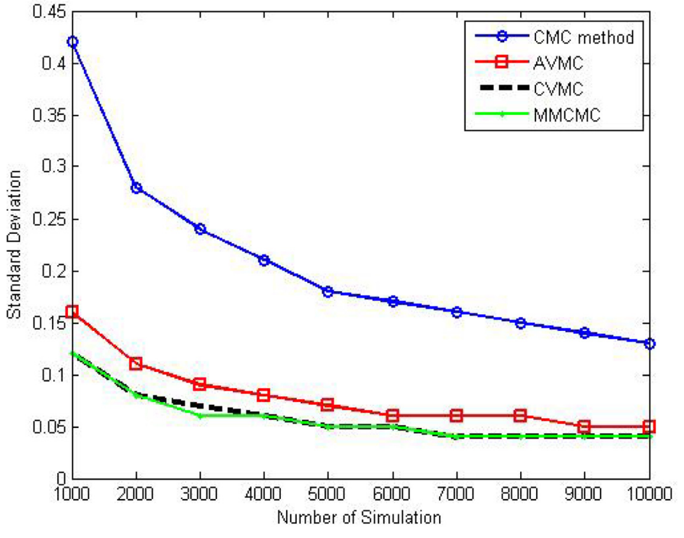

Figure 1. Standard deviations of European basket option price for n=4.

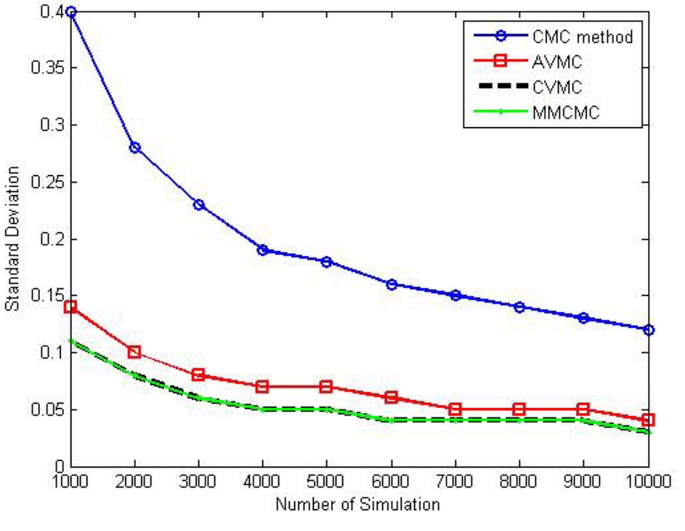

Figure 2. Standard deviations of European basket option price for n=10.

Example 2. We consider the actual spread option on 4 assets of S&P 500 index[2] regarding to four companies, Microsoft, Intel, Oracle, and Cisco, from January 2012 to December, 2012. We set

$$K = 23, r = 0.01,\ T = 1, w_i = \frac{1}{4} \quad (i = 1,\ldots,4)..$$

The initial asset prices are S_{01} = 25.87, S_{01} = 26.77, S_{01} = 24.54, S_{01} = 18.63 and the volatilities of 4 assets (per one year) are σ_{01} = 0.204, σ_{02} =0.207, σ_{03} =0.211, σ_{04} = 0.258, respectively. The correlation matrix for the asset returns is given by

[2] Source: http://finance.yahoo.com

$$\Sigma = \begin{pmatrix} 1 & 0.55 & 0.53 & 0.51 \\ 0.55 & 1 & 0.55 & 0.48 \\ 0.53 & 0.55 & 1 & 0.47 \\ 0.51 & 0.48 & 0.47 & 1 \end{pmatrix}$$

Table 3 reports the option prices and the SD errors for each of the four methods by 10,000 replications of the Monte Carlo simulation. Looking at the SD errors, we conclude that the MMC technique is extremely efficient and more accurate than existing techniques. Moreover, Figures 3 and 4 confirms the Black-Scholes assumptions for the actual data.

Table 3. Simulation results for European basket option.

N	Crude MC		AV		CV		MMC	
	$\hat{C}$	SD	$\hat{C}$	SD	$\hat{C}$	SD	$\hat{C}$	SD
10000	2.287	0. 030	2.264	0.015	2.260	0.011	2.258	0.008

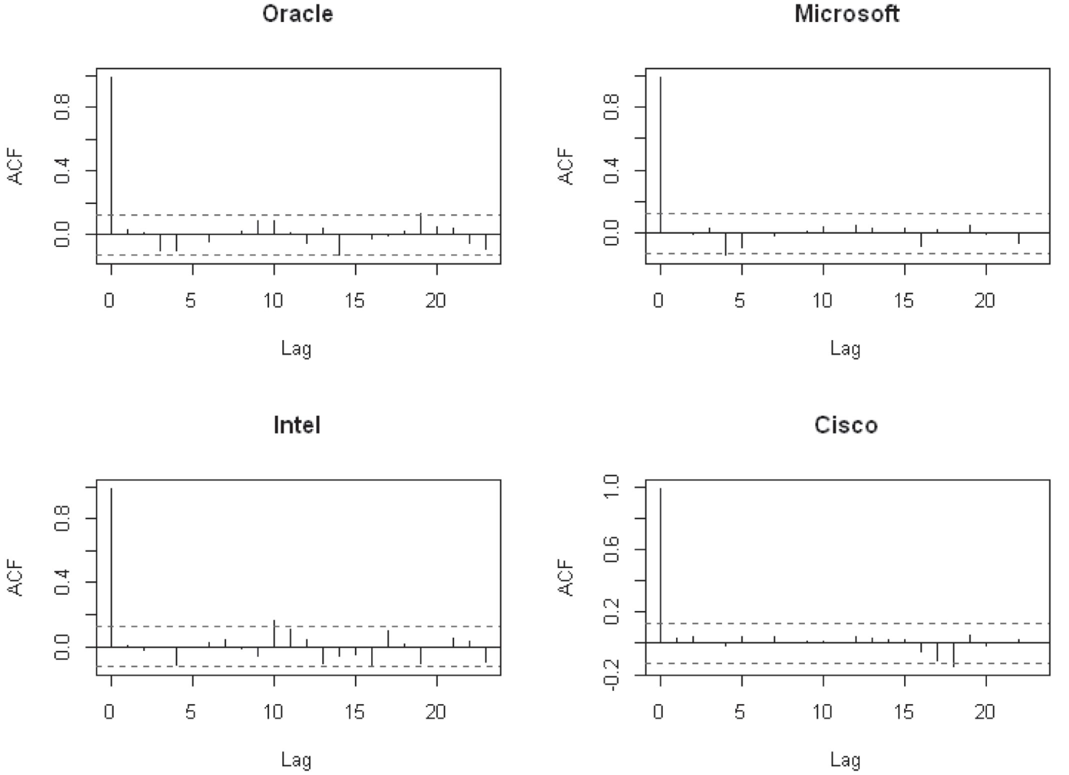

Figure 3. ACF chart regarding to four companies.

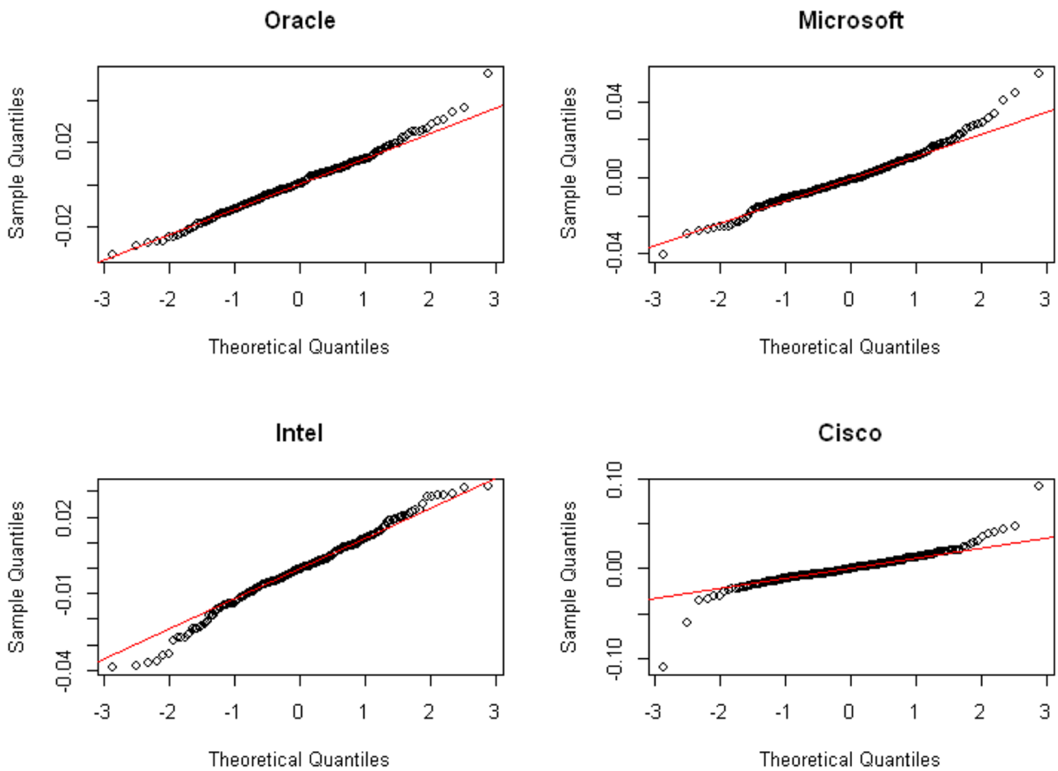

Figure 4. Quantile-quantile plot regarding to four companies.

5.CONCLUSION

Monte Carlo simulations often used to price options that have no closed- form. To increase the accuracy of Monte Carlo method there are a variety of methods [13]. In this paper, antithetic variates, control variates and Pellizzari's (MMC) methods were used to price European basket option. Based on the simulation results, control variates perform better than the antithetic variates. Also, we have shown that the performance of Pellizzari's method (MMC) to construct new control variates is very similar to standard control variates when the number of underlying assets increases.

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