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Contributed Paper

Finite-dimensional Simple Poisson Modules

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ABSTRACT

We classify the finite-dimensional simple Poisson modules for two Poisson algebras. The first is related to the invariants for an automorphism of the torus and to the cyclically q-deformed algebra $U'_q(so_3)$ of [1,2]. We find that there are five d-dimensional simple Poisson modules for each $d \geq 1$. The second is the Poisson algebra arising from the quantized enveloping algebra $U_q(sl_2)$ using a presentation discovered by Ito, Terwilliger and Weng [3] and we find that there are two d-dimensional simple Poisson modules for each $d \geq 1$.

Keywords: poisson algebras, poisson ideal, poisson maximal ideal and poisson automorphism

1. INTRODUCTION

A *Poisson algebra* is a commutative $\mathbb{C}$-algebra equipped with a Lie bracket $\{-,-\}$ for which $\{a,-\}: A \to A$ is always a derivation of A. The bracket $\{-,-\}$ is then called a *Poisson bracket*. A *Poisson ideal* is an ideal of A for both the algebra structures on A.

If T is a $\mathbb{C}$-algebra with a central non-unit non-zero-divisor t such that $A := T/tT$ is commutative then there is a Poisson bracket $\{-,-\}$ on A such that $\{\bar{x}, \bar{y}\} = \overline{t^{-1}[x,y]}$ for all $\bar{x}, \bar{y} \in A$. In this situation, we shall follow [4, III 5.4] in referring to T as a quantization of the Poisson algebra A.

We consider Poisson A–modules in the senses of [5] and classify the finite-dimensional simple Poisson modules for two examples of interest. The first is the Poisson bracket on $A = \mathbb{C}[x, y, z]$ with

$$\begin{aligned} \{x, y\} &= yx + z; \\ \{y, z\} &= zy + x; \\ \{z, x\} &= xz + y. \end{aligned} \tag{1.1}$$

This arises in [6] in connection with automorphisms of the co-ordinate rings of the torus and quantum torus and it has a quantization, in the above sense, that is isomorphic to the cyclically q-deformed algebra $U'_q(so_3)$ of [1,2]. The second has a quantization which gives the quantized enveloping algebra $U_q(sl_2)$ in the equitable presentation discovered by Ito, Terwilliger and Weng [3]. The results presented here, classifying the finite-dimensional simple Poisson modules for these two Poisson algebras by a direct method based on Kassel [7, Theorem V.4.4], appeared in the author's thesis [8]. They have also been classified by a different method by Jordan [9].

We shall show, in Section 3, that the annihilator of any finite-dimensional simple Poisson A-module is a Poisson maximal ideal. To classify finite-dimensional simple Poisson A-modules, we first identify the Poisson maximal ideals. We then analyse the Poisson modules annihilated by each Poisson maximal ideal in turn. We shall present, in Section 4, full details for one Poisson maximal ideal of the first example and then indicate the details of the changes needed in other cases.

2. Definitions and Notation

Throughout A will be a finitely generated commutative algebra over $\mathbb{C}$.

Definition 2.1 A *Poisson bracket* on A is a Lie algebra bracket $\{-,-\}$ satisfying the Leibniz rule $\{ab, c\} = a\{b, c\}+\{a, c\}b$ for all $a,b,c \in A$. The pair $(A, \{-,-\})$ is called a *Poisson algebra*.

A subalgebra B of A is a *Poisson subalgebra* of A if $\{b, c\} \in B$ for all $b, c \in B$ and an ideal I of A is a *Poisson ideal* if $\{i, a\} \in I$ for all $i \in I$ and all $a \in A$. If I is a Poisson ideal of A then A/I is a Poisson algebra in the obvious way: $\{a + I, b + I\} = \{a, b\} + I$. A Poisson algebra A is said to be *simple* if its only Poisson ideals are (0) and A.

Definition 2.2. Let P be an ideal of a Poisson algebra A. Then P is a *Poisson prime ideal* if P is both a prime ideal and a Poisson ideal. It follows from [10, 3.3.2] that this is equivalent to saying that P is a Poisson ideal and, for all Poisson ideals $I,J \subseteq A$,

$IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

Definition 2.3. By *maximal Poisson ideal*, we shall mean a Poisson ideal I of A such that if J is a Poisson ideal and $I \not\subseteq J$ then $J = A$. An ideal I of a Poisson algebra A is said to be a *Poisson maximal ideal* if I is a maximal ideal of A and also a Poisson ideal. For example, let $A = \mathbb{C}[x,y]$ which is a Poisson algebra with the Poisson bracket $\{x,y\} = 1$. Then 0 is a maximal Poisson ideal but is not a Poisson maximal ideal.

Definition 2.4. Let R be a commutative $\mathbb{C}$-algebra and let $h \in R$. Let A be an R-algebra and suppose that h is not a zero divisor in A, and that $\bar{A} := A/hA$ is a commutative $\mathbb{C}$-algebra. Then there is a Poisson bracket $\{-,-\}$ on $\bar{A}$ such that $\{\bar{a}, \bar{b}\} = \overline{h^{-1}[a,b]}$ for all $\bar{a} = a + hA$ and $\bar{b} = b + hA$. Following [4, III.5.4], we call A a *quantization* of the Poisson algebra $\bar{A}$.

There is more than one definition of Poisson module the literature. We shall use the one introduced by D.R. Farkas [5].

Definition 2.5. Let A be a commutative Poisson algebra with Poisson bracket $\{-,-\}$. We shall say that an A-module M is a *Poisson module* if there is a bilinear form $\{-,-\}_M : A \times M \to M$ such that

(i) $\{a, a'm\}_M = \{a, a'\}m + a'\{a, m\}_M$;

(ii) $\{aa', m\}_M = a\{a', m\}_M + a'\{a, m\}_M$;

(iii) $\{\{a, a'\}, m\}_M = \{a, \{a', m\}_M\}_M - \{a', \{a, m\}_M\}_M$;

for all $a, a' \in A$ and all $m \in M$.

A submodule N of a Poisson module M is called a *Poisson submodule* if $\{a, n\}_M \in N$, for all $a \in A$, $n \in N$.

Definition 2.6. Let N be a left module over a ring R. Give any subset $X \subseteq N$, the annihilator of X is the set $\mathrm{ann}_R(X) = \{r \in R: rx = 0$ for all $x \in X\}$, which is a left ideal of R.

Lemma 2.7. Let A be a Poisson algebra and M be a Poisson A - module.

(i) The annihilator $\mathrm{ann}_A(M)$ is a Poisson ideal of A;

(ii) if M is a simple Poisson module then $\mathrm{ann}_A(M)$ is a Poisson prime ideal of A;

(iii) if M is a finite-dimensional simple Poisson module then $\mathrm{ann}_A(M)$ is a Poisson maximal ideal of A.

Proof. (i) Let $a'M = 0$ for some $a' \in A$. It follows from Definition 2.5(i) that $0 = \{a,a'm\}_M = \{a,a'\}m$ for all $a \in A$ so $\{a,a'\} \in \mathrm{ann}_A(M)$. It follows that $\mathrm{ann}_A(M)$ a Poisson ideal of A.

(ii) Let M be a simple Poisson module and let I and J be Poisson ideals of A. Let $P = \mathrm{ann}_A(M)$. Suppose that $IJ \subseteq P$, that is, $IJM = 0$. We show that JM is a Poisson submodule of M. Let $j \in J$ and $m \in M$. For $a \in A$, $\{a,jm\}_M = j\{a,m\}_M + \{a, j\}m \in JM$. Hence JM is a Poisson submodule of M. Since M is a simple Poisson module, $JM = 0$ or $JM = M$. If $JM = M$ then $IM = IJM = 0$, so $J \subseteq P$ or $I \subseteq P$. This shows that P is a Poisson prime ideal of A.

(iii) Let M be a finite-dimensional simple Poisson module and $P = \mathrm{ann}_A(M)$. By (ii), P is a Poisson prime ideal of A. Then M is a faithful A/P- module. Let θ be the map from A/P to the endomorphism ring of M, $End_{\mathbb{C}}(M)$, given by $\theta(a + P)(m) = am$ for $a \in A$ and $m \in M$. We claim that θ is an injective $\mathbb{C}$- homomorphism. It is clear that θ well-defined.

Let $a, b \in A$. We see that $\theta(\bar{a}\, \bar{b})(m) = abm = \theta(\bar{a})\, bm = \theta(\bar{a})\theta(\bar{b})(m)$.

Therefore $\theta(\bar{a}\bar{b}) = \theta(\bar{a})\theta(\bar{b})$, whence θ is a $\mathbb{C}$-homomorphism. Let $\bar{a} \in A/P$ be such that $\theta(\bar{a}) = 0$. Then $am = 0$ for all $m \in M$ and $a \in P$ which implies $\ker \theta = 0$. Hence θ is injective. As $\dim_{\mathbb{C}}(M) < \infty$, $\dim_{\mathbb{C}}(A/P) \le \dim_{\mathbb{C}} End_{\mathbb{C}}(M) = (\dim_{\mathbb{C}}(M))^2$. Since A/P is a finite-dimensional algebra over $\mathbb{C}$, it follows that A/P is an Artinian ring. We also know that A/P is a prime ring because P is a prime ideal. Therefore A/P is simple, whence P is maximal.

Lemma 2.8. Let $A = \mathbb{C}[x_1, x_2,..., x_n]$ with a Poisson bracket $\{-,-\}$. Let $V = \mathrm{Sp}(x_1, x_2,..., x_n)$ and let M be an A-module. Suppose that there is a bilinear form $\{-,-\}_M : V \times M \to M$. Extend this to a bilinear form $\{-,-\}_M : A \times M \to M$ using Definition 2.5(ii) and $\{1, m\}_M = 0$. If Definition 2.5(i) and (iii) hold, for all $m \in M$, whenever $a = x_i$, and $a' = x_j$ for $1 \le i < j \le n$ then Definition 2.5(i) and (iii) hold for all $a, a' \in A$.

Proof. The extension of $\{-,-\}_M$ from $V \times M$ to $A \times M$, using Definition 2.5(ii), is well-defined because A is free as a commutative algebra and is such that, for example,

$$\{x_1x_2 \cdots x_n, m\}_M = \sum_{l=1}^{n} x_1x_2 \cdots \hat{x_l} \cdots x_n \{x_l, m\}_M$$

where $\hat{x}_l$ denotes omission of x_l. If Definition 2.5(i) and (iii) hold whenever $a = x_i$ and $a' = x_j$, $1 \le i < j \le n$, they hold whenever $a = x_i$ and $a' = x_j$ for any i and j because $\{x_i, x_i\} = 0$ and $\{x_i, x_j\} = -\{x_j, x_i\}$. Let $a \in A$. Let $L(a') = \{a \in A$: (i) holds for all $m \in M\}$ and $L = \{a' \in A$: (i) holds for all $m \in M$, $a \in A\}$. For $r, s \in L(a')$ and $m \in M$, it is not difficult to check that $\{rs, a'm\}_M = \{rs,a'\}m + a'\{rs,m\}_M$ and $\{a, rsm\}_M = \{a, rs\}m + rs\{a,m\}_M$.

Therefore $rs \in L(a')$. By bilinearity, $L(a')$ is a subspace of A so $L(a')$ is also a subalgebra of A because the Poisson bracket $\{-,-\}$ is a bilinear form.

Let $R(a) = \{a' \in A$: (iii) holds for all $m \in M\}$ and $R = \{a \in A$: (iii) holds for all $m \in M, a' \in A\}$. For $r, s \in R(a)$ and $m \in M$, it is a routine matter to show that $\{\{rs, a'\},m\}_M = \{rs, \{a',m\}_M\}_M - \{a', \{rs,m\}_M\}_M$. Therefore $rs \in R(a)$. By bilinearity, $R(a)$ is a subspace of A, so $R(a)$ is a subalgebra of A. Similarly, R is a subalgebra of A.

As $L(a)$ and $R(a)$ are subalgebras containing x_i for all x_i, $1 \leq i \leq n$ we have $R(a) = A = L(a)$. Hence R and L are also subalgebras containing x_i for all i, $1 \leq i \leq n$, so $R = A = L$.

3. Poisson Automorphisms

Let $(R, \{-,-\})$ be a Poisson algebra. We say that a $\mathbb{C}$-algebra automorphism α: $R \rightarrow R$ is a Poisson automorphism if for all $x, y \in R$, $\alpha(\{x, y\}) = \{\alpha(x), \alpha(y)\}$.

Poisson automorphisms can be used to twist the module structures of a Poisson modules, as specified in the following theorem.

Theorem 3.1 Let A be a Poisson algebra, let α be a Poisson automorphism of A and let M be a Poisson module. Define $a.m = \alpha(a)m$ and $\{a,m\}^{\alpha}_M = \{\alpha(a),m\}_M$ for all $a \in A$ and $m \in M$. Then M is a Poisson module, which we denote M^{α}, under $-.-$: $A \times M \rightarrow M$ and $\{-,-\}^{\alpha}_M : A \times M \rightarrow M$.

Proof. It is well-known that M is an A-module under $-.-$ see [4], so it suffices to check axioms (i), (ii) and (iii) from 2.5 as follows. Let $a, a' \in A$ and $m \in M$. Then, for (i),

$$\begin{aligned}\{a, a'.m\}^{\alpha}_M &= \{\alpha(a),a'.m\}_M \\ &= \{\alpha(a),\alpha(a')m\}_M \\ &= \{\alpha(a),\alpha(a')\}m + \alpha(a')\{\alpha(a), m\}_M \\ &= \alpha\{a, a'\}m + \alpha(a')\{a, m\}^{\alpha}_M \\ &= \{a, a'\}.\ m + a'.\{a, m\}^{\alpha}_M .\end{aligned}$$

Similar calculations show

(ii) $\{aa',m\}^{\alpha}_M = a.\{a',m\}^{\alpha}_M + a'.\{a,m\}^{\alpha}_M$

and

(iii) $\{\{a,a'\},m\}^{\alpha}_M = \{a, \{a',m\}^{\alpha}_M\}^{\alpha}_M - \{a', \{a, m\}^{\alpha}_M\}^{\alpha}_M$.

Thus M^{α} is a Poisson module.

Remark 3.2 Let M be a Poisson A-module and let α be a Poisson automorphism of A. Then the Poisson submodules of M^{α} have the form N^{α} where N is a Poisson submodule of M and hence that is a simple Poisson module if and only if M^{α} is a simple Poisson module. Also if $J = \text{ann}_A M$ is the annihilator of M then $\text{ann}_A M^{\alpha} = \alpha^{-1}(J)$.

4. Examples of Poisson Modules

In this section, we determine the finite-dimensional simple modules for two interesting examples.

Example 4.1 Let T be the $\mathbb{C}$-algebra generated by x, y, z, t and t^{-1}, with t central, and the three relations

$$\begin{aligned} xy - tyx &= (t-1)z; \\ yz - tzy &= (t-1)x; \\ zx - txz &= (t-1)y. \end{aligned} \tag{4.1}$$

If $q \in \mathbb{C}\backslash\{1\}$ then $T/(t-q)T$ is isomorphic to the algebra the cyclically q – deformed algebra $U'_q(so_3)$ of [1, 2].

Let $A := T/(t-1)T \simeq \mathbb{C}[x, y, z]$ which is a commutative polynomial algebra. The induced Poisson bracket on A is such that

$$\begin{aligned} \{x, y\} &= yx + z; \\ \{y, z\} &= zy + x; \\ \{z, x\} &= xz + y. \end{aligned}$$

Here we are abusing notation by writing x, y and z for both elements of T and their images in A. This Poisson bracket on A is related to invariants of the Poisson automorphism α of $\mathbb{C}[x_1^{\pm1}, x_2^{\pm1}]$, where $\{x_1, x_2\} = x_1 x_2$, $\alpha(x_1) = x_1^{-1}$ and $\alpha(x_2) = x_2^{-1}$.

If we write $u_1 = -2x$, $u_2 = -2y$ and $u_3 = -2z$ then

$$\{u_1, u_2\} = u_2u_1 - 2u_3$$
$$\{u_2, u_3\} = u_3u_2 - 2u_1$$
$$\{u_3, u_1\} = u_1u_3 - 2u_2$$

and the Poisson algebra of invariants for α has the form A/I for the Poisson ideal $I = (u_1u_2u_3 - u_1^2 - u_2^2 - u_3^2 + 4)A$ of A. See [6, Examples 3.3 and 3.6] for more detail.

Let J be a Poisson maximal ideal of A. Since A is a commutative polynomial ring over $\mathbb{C}$, $J = (x - a, y - b, z - c)$ for some $a, b, c \in \mathbb{C}$. As J is Poisson, $\{x, J\} \subseteq J$, $\{y, J\} \subseteq J$, and $\{z, J\} \subseteq J$. Observe that $yx + z = \{x, y - b\} \in J, -(zx + y) = \{x, z - c\} \in J$, $zy + x = \{y, z - c\} \in J$. This happens precisely when $ab + c = ac + b = bc + a = 0$. As $c = -ab$, we have

$0 = b - a^2 b = a - ab^2$. This implies that $a = \pm 1$ or $b = 0$. Similarly $c = 0$ or $b = \pm 1$ and $a = 0$ or $c = \pm 1$. If $b = 0$ then $a = c = 0$ and, similarly, if $a = 0$ or $c = 0$ then $a = b = c = 0$. There are only five solutions, so there are precisely five Poisson maximal ideals:

$$J_1 = xA + yA + zA$$
$$J_2 = (x + 1)A + (y + 1)A + (z + 1)A$$
$$J_3 = (x + 1)A + (y - 1)A + (z - 1)A$$
$$J_4 = (x - 1)A + (y + 1)A + (z - 1)A$$

and

$$J_5 = (x - 1)A + (y - 1)A + (z + 1)A.$$

The annihilator of any finite-dimensional Poisson A-module must be one of these. We next classify finite-dimensional Poisson module annihilated by J_1.

Lemma 4.2 Let M be a Poisson module annihilated by $J_1 = xA + yA + zA$ and let $m \in M$. Then we have:

(i) $xm = ym = zm = 0.$

(ii) $\{yx,m\}_M = \{zy,m\}_M = \{zx,m\}_M = 0.$

(iii)(a) $\{z,m\}_M = \{x, \{y,m\}_M\}_M - \{y, \{x,m\}_M\}_M$;

(b) $\{x,m\}_M = \{y, \{z,m\}_M\}_M - \{z, \{y,m\}_M\}_M$;

(c) $\{y,m\}_M = \{z, \{x,m\}_M\}_M - \{x, \{z,m\}_M\}_M$.

Proof. This is a routine calculation.

Remark 4.3 Let M be a Poisson module annihilated by J_1. Let $m \in M$ be an eigenvector for $\{x,-\}_M$ with eigenvalue $\lambda \in \mathbb{C}$. Thus $\{x,m\}_M = \lambda m$. It follows from Lemma 4.2 (iii) that

$$\{x, \{y,m\}_M\}_M = \{z,m\}_M + \lambda\{y,m\}_M \tag{4.3}$$

$$\{x, \{z,m\}_M\}_M = \lambda\{z,m\}_M - \{y,m\}_M \tag{4.4}$$

$$\{y, \{z,m\}_M\}_M - \{z, \{y,m\}_M\}_M = \{x,m\}_M = \lambda m. \tag{4.5}$$

To simplify these, let $u := \frac{1}{2}(y-iz)$ and $v := \frac{1}{2}(z-iy)$ and $J_1 = xA + uA + vA$.
It is a routine matter to check that $u,v,$ and x generate A so $A = \mathbb{C}[x,u,v]$ with the Poisson bracket

$$\{x,v\} = -ixu - iv,$$
$$\{x,u\} = ixv + iu,$$
$$\{u,v\} = \tfrac{1}{2}(x + i(u^2 + v^2)). \tag{4.6}$$

It follows from (4.3), (4.4) and (4.5) that

$$\{x, \{v,m\}_M\}_M = (\lambda - i)\{v,m\}_M, \tag{4.7}$$

$$\{x, \{u,m\}_M\}_M = (\lambda + i)\{u,m\}_M, \tag{4.8}$$

$$\{u, \{v,m\}_M\}_M - \{v, \{u,m\}_M\}_M = \tfrac{1}{2}\lambda m. \tag{4.9}$$

Lemma 4.4 Let $A = \mathbb{C}[x,u,v]$ with the Poisson bracket as in (4.6). Let $d \geq 1$. There is a d-dimensional Poisson A-module M, with basis $\{m_1,m_2,...,m_d\}$, such that $xM = vM = uM = 0$ and

(i) $\{x,m_j\}_M = (\lambda + (j - 1)i)m_j$ for $1 \leq j \leq d$;

(ii) $\{v,m_1\}_M = 0$ and $\{v,m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ for $1 < j \leq d$;

(iii) $\{u,m_j\}_M = m_{j+1}$ for $1 \leq j < d$ and $\{u,m_d\}_M = 0$, where $\lambda = \frac{1-d}{2} i$.

Proof. Let M be a d-dimensional vector space with basis $\{m_1,m_2,...,m_d\}$. For $1 \le j \le d$, set $xm_j = um_j = vm_j = 0$ and define $\{x,m_j\}_M$, $\{u,m_j\}_M$ and $\{v,m_j\}_M$ in accordance with (i), (ii) and (iii). It is a routine matter to check that Definition 2.5 (i) and (iii) hold for $m = m_j$, for $1 \le j \le d$, when the pair $\{a,a'\}$ is (x,u), (x, v) or (u, v). We then extend the Poisson action on M from $V := \mathbb{C}x + \mathbb{C}u + \mathbb{C}v$ to $\mathbb{C}[x,u,v]$ using Definition 2.5(ii). Thus M becomes a Poisson module with the stated properties.

Lemma 4.5 Let $d \ge 1$. The d-dimensional Poisson module constructed in Lemma 4.4 is simple as a Poisson module.

Proof. Let $\lambda_j = \lambda + (j-1)i$, $1 \le j \le d$. Note that $\lambda_j \ne \lambda_k$ when $j \ne k$. Let N be a non-zero Poisson submodule of M. Let $0 \ne n = \sum_{j=1}^{d} \alpha_j m_j \in N$ be such that minimally many of the coefficients $\alpha_j \in \mathbb{C}$ are non-zero and choose k so that $\alpha_k \ne 0$. Consider the element

$\{x,n\}_M - \lambda_k n = \sum_{j=1}^{d} \alpha_j(\lambda_j - \lambda_k)m_j$. This has one fewer non-zero coefficient than n so, by minimality, it is 0 and hence $\alpha_j = 0$ when $j \ne k$, that is $n = \alpha_k m_k$. Therefore $m_k \in N$. By the Poisson action of u and v, $m_j \in N$ for all j. So $N = M$ and M is a simple Poisson module.

Lemma 4.6 Let M be a finite-dimensional simple Poisson module annihilated by $J_1 = uA + vA + xA$ and let $n \le dim_{\mathbb{C}} M$. There exist $\lambda \in \mathbb{C}$ and n linearly independent elements $m_1, m_2 ..., m_n \in M$ such that

(i) $\{x,m_j\}_M = (\lambda + (j-1)i)m_j$ for $1 \le j \le n$;

(ii) $\{v,m_1\}_M = 0$ and $\{v,m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ for $1 < j \le n$;

(iii) $\{u,m_j\}_M = m_{j+1}$ for $1 \le j < n$.

Proof. Let $\Lambda = \{\lambda \in \mathbb{C} : \{x,m\}_M = \lambda m$ for some $0 \ne m \in M\}$. Since $dim_{\mathbb{C}} M < \infty$ the linear transformation of M, with $m \to \{x,m\}_M$, has an eigenvalue, therefore $\Lambda \ne \varnothing$. Let $\lambda \in \Lambda$. As x,u,v generate A and it follows from (4.7) and (4.8) that $\{m \in M : \{x,m\}_M = (\lambda + ni)m$ for some $n \in \mathbb{Z}\}$ spans a non-zero Poisson module of M. Since M is finite-dimensional, $\lambda \in \Lambda$ can be chosen so that $\lambda - i \notin \Lambda$. Let m_1 be an eigenvector for $\{x,-\}_M$ with eigenvalue λ. Since $\{x,\{v, m_1\}_M\}_M = (\lambda - i)\{v, m_1\}_M$, $\{v, m_1\}_M = 0$. Thus the result is true when $n = 1$. We proceed by induction on n. Suppose that (i), (ii) and (iii) hold for n and that $n+1 \le dim_{\mathbb{C}} M$. Then $\{u, m_n\}_M \ne 0$, otherwise $\mathrm{Sp}(m_1,\ldots,m_n)$ is an n-dimensional Poisson submodule of M, contrary to the Poisson simplicity of M. Let $m_{n+1} = \{u, m_n\}_M$ in accordance with (i). By (4.8) $\{x, m_n\}_M = (\lambda + (n-1)i)m_n$ and $\{x,m_{n+1}\}_M = \{x, \{u,m_n\}_M\}_M = (\lambda + ni)m_{n+1}$, in accordance with (iii). For (ii), observe that

$$\{u, \{v, m_n\}_M - \{v, \{u,m_n\}_M\}_M = \tfrac{1}{2}\{x, m_n\}_M,$$

so

$$-\tfrac{1}{2}(n-1)(\lambda + \tfrac{1}{2}(n-2)i)\{u, m_{n-1}\}_M - \{v,m_{n+1}\}_M = \tfrac{1}{2}(\lambda + (n-1)i)m_n.$$

and hence,

$$\{v, m_{n+1}\}_M = -\tfrac{1}{2}(n-1)(\lambda + \tfrac{1}{2}(n-2)i)m_n - \tfrac{1}{2}(\lambda + (n-1)i)m_n.$$

It follows that

$$\{v, m_{n+1}\}_M = -\tfrac{1}{2}n(\lambda + \tfrac{1}{2}(n-1)i)m_n.$$

Note that, being eigenvectors for $\{x,-\}_M$ with distinct eigenvalues, $m_1, m_2,\ldots,m_{n+1}$ are linearly independent. The result holds by induction on n.

Theorem 4.7 Let M be a finite-dimensional simple Poisson module annihilated by $J_1 = xA + uA + vA$ and let $d = dim_{\mathbb{C}} M$. There exist d linearly independent elements $m_1, m_2,\ldots,m_d \in M$ such that

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ for $1 \le j \le d$;

(ii) $\{v,m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ for $1 \le j \le d$;

(iii) $\{u, m_j\}_M = m_{j+1}$ for $1 \le j < d$, and $\{u, m_d\}_M = 0$, where $\lambda = -\frac{d-1}{2}i$.

Proof. By Lemma 4.6, there exist $\lambda \in \mathbb{C}$ and linearly independent elements $m_1, m_2,\ldots,m_d$ of M such that

(i) $\{x, m_j\}_M = (\lambda + (j-1)i)m_j$ for $1 \le j \le d$;

(ii) $\{v,m_j\}_M = -\frac{1}{2}(j-1)(\lambda + \frac{1}{2}(j-2)i)m_{j-1}$ for $1 \le j \le d$ and

(iii) $\{u, m_j\}_M = m_{j+1}$ for $1 \le j < d$.

As $\dim_{\mathbb{C}} M = d$, $M = Sp(m_1,m_2,\ldots,m_d)$ and is the sum of the eigenspaces for $\{x,-\}_M$ and the eigenvalues $\lambda, \lambda + i, \ldots \lambda + (d-1)i$. By (4.8),

$$\{x, \{u, m_d\}_M\}_M = (\lambda + di)\{u, m_d\}_M$$

but $\lambda + di$ is not an eigenvalue of $\{u, m_d\}_M$ for $\{x, -\}_M$ so $\{u, m_d\}_M = 0$. By (4.9),

$$\begin{aligned}\{u, \{v,m_d\}_M\}_M - \{v, \{u, m_d\}_M\}_M &= \tfrac{1}{2}\{x, m_d\}_M \\ -\tfrac{1}{2}\left(\lambda+\tfrac{1}{2}(d-2)i\right)\{u, m_{d-1}\}_M &= \tfrac{1}{2}(\lambda + (d-1)i)m_d \\ -\tfrac{1}{2}\left(\lambda+\tfrac{1}{2}(d-2)i\right)m_d &= \tfrac{1}{2}(\lambda + (d-1)i)m_d \\ d(\lambda+\tfrac{1}{2}(d-1)i) &= 0\end{aligned}$$

from which it follows that $\lambda = -\frac{d-1}{2}i$.

We conclude from Lemma 4.4 and Theorem 4.7 that for $d \ge 1$ there is a unique d-dimensional simple Poisson module M annihilated by J_1 as the following theorem.

Theorem 4.8 Let $d \ge 1$. There is a unique d-dimensional simple Poisson module over A, annihilated by J_1. It has a basis $m_1, m_2,\ldots,m_d$ such that

(i) $\{x,m_j\}_M = (\lambda + (j-1)i)m_j$ for $1 \le j \le d$;

(ii) $\{y,m_1\}_M = m_2$, $\{y,m_j\}_M = m_{j+1} - \frac{1}{2}i(j-1)(\lambda+\frac{1}{2}(j-2)i)m_{j-1}$ for $1 < j < d$ and $\{y, m_d\}_M = -\frac{1}{2}i(d-1)(\lambda+\frac{1}{2}(d-2)i)m_{d-1}$,

(iii) $\{z,m_1\}_M = im_2$, $\{z,m_j\}_M = im_{j+1} - \frac{1}{2}(j-1)(\lambda+\frac{1}{2}(j-2)i)m_{j-1}$ for $1 < j < d$, $\{z,m_d\}_M = -\frac{1}{2}(d-1)(\lambda+\frac{1}{2}(d-2)i)m_{d-1}$,

where $\lambda = \frac{1-d}{2}i$.

Proof. This is immediate from Lemma 4.4 and Theorem 4.7.

We next consider Poisson modules annihilated by $J_2 = (x+1)A + (y+1)A + (z+1)A$.

Lemma 4.9 Let M be a Poisson module annihilated by

$$J_2 = (x+1)A + (y+1)A + (z+1)A$$

and let $m \in M$. Then we have :

(i) $xm = ym = zm = -m$.

(ii) (a) $\{xy,m\}_M = -\{y,m\}_M - \{x,m\}_M$;

(b) $\{yz,m\}_M = -\{y,m\}_M - \{z,m\}_M$;

(c) $\{xz,m\}_M = -\{z,m\}_M - \{x,m\}_M$.

(iii) (a) $\{x,\{y,m\}_M\}_M - \{y,\{x,m\}_M\}_M = \{z,m\}_M - \{x,m\}_M - \{y,m\}_M$;

(b) $\{y,\{z,m\}_M\}_M - \{z,\{y,m\}_M\}_M = \{x,m\}_M - \{y,m\}_M - \{z,m\}_M$;

(c) $\{z,\{x,m\}_M\}_M - \{x,\{z,m\}_M\}_M = \{y,m\}_M - \{x,m\}_M - \{z,m\}_M$.

Proof. This is a routine calculation.

The method used to classify finite-dimensional simple Poisson module annihilated by J_1 can be applied to classify those annihilated by J_2. Here we change generators to x, y and $u := z - x - y$ so that $J_2 = (x+1)A + (y+1)A + (u-1)A$ and the Poisson bracket is:

$$\begin{aligned}\{x, y\} &= (x+1)y + u + x, \\ \{x, u\} &= -(x+1)(2y + u + x), \\ \{y, u\} &= (y+1)(y + u + 2x). \end{aligned} \qquad (4.10)$$

Adapting the proofs of Lemmas 4.2, 4.4, 4.5 and 4.6 and Theorems 4.7 and 4.8, we obtain the following theorem.

Theorem 4.10. Let $d \geq 1$. There is a unique d-dimensional simple Poisson module over A, annihilated by J_2. It has a basis m_1, $m_2, \ldots, m_d$ such that

(i) $\{x, m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1}$ for $1 \leq j \leq$ d;

(ii) $\{y, m_j\}_M = m_{j+1}$ for $1 \leq j <$ d, and $\{y, m_d\}_M = 0$,

(iii) $\{z, m_j\}_M = (j-1)(\lambda + j - 2)m_{j-1} + (\lambda + 2(j-1))m_j + m_{j+1}$, for $1 \leq j <$ d, and $\{z, m_d\}_M = (d-1)(\lambda + d - 2)m_{d-1} + (\lambda + 2(d-1))m_d$,

where $\lambda = 1 - d$.

To classify the simple Poisson modules annihilated by J_3, J_4 and J_5, we use the simple Poisson modules annihilated by J_2 twisted by Poisson automorphisms. Recall that the Poisson bracket of A is

$$\{x, y\} = yx + z,$$
$$\{y, z\} = zy + x,$$
$$\{z, x\} = xz + y. \qquad (4.11)$$

Let α, β and γ be the $\mathbb{C}$-automorphisms of A such that

(i) $\alpha(x) = x, \alpha(y) = -y, \ \alpha(z) = -z$,

(ii) $\beta(x) = -x, \beta(y) = y, \ \beta(z) = -z$,

(iii) $\gamma(x) = -x, \gamma(y) = -y, \ \gamma(z) = z$.

Then we can check that α, β and γ are Poisson automorphisms of R. Observe that $\alpha(J_2) = J_3$, $\beta(J_2) = J_4$ and $\gamma(J_2) = J_5$. As $\alpha^2 = \beta^2 = \gamma^2 = id$, the simple Poisson modules annihilated by J_3 are precisely the Poisson modules M^α where M is a simple Poisson module annihilated by J_2. By using the same method of simple Poisson modules annihilated by J_2, we can conclude that for each $d \geq 1$ there is precisely one d-dimensional simple Poisson module annihilated by J_3.

Using β and γ in place of α respectively, we obtain the same conclusion for J_4 and J_5.

Theorem 4.11 For $d \geq 1$, the Poisson algebra A has precisely five d-dimensional simple Poisson modules.

Proof. This is immediate from Theorem 4.8, Theorem 4.10 and using the Poisson automorphisms of A.

Our second example arises from the quantized enveloping algebra $U_q(sl_2)$ using a presentation discovered by Ito, Terwilliger and Weng [3].

Example 4.12 Let $q \neq \pm 1$. The quantized enveloping algebra $U_q(sl_2)$ has a presentation [3] with generators $x^{\pm 1}$, y, z and relations $xx^{-1} = x^{-1}x = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad (4.12)$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \qquad (4.13)$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \qquad (4.14)$$

Alternatively, the relations may be written as

$$qxy - q^{-1} yx = q - q^{-1}, \qquad (4.15)$$
$$qyz - q^{-1} zy = q - q^{-1}, \qquad (4.16)$$
$$qzx - q^{-1} xz = q - q^{-1}. \qquad (4.17)$$

If $q = 1$ then these become

$$xy - yx = 0, \ yz - zy = 0, \ zx - xz = 0. \quad (4.18)$$

Let T be the $\mathbb{C}$-algebra with the generators x, x^{-1}, y, z and $t^{\pm 1}$ subject to the relations

$$xy - t^{-2} yx = 1 - t^{-2}, \qquad (4.19)$$
$$yz - t^{-2} zy = 1 - t^{-2}, \qquad (4.20)$$
$$zx - t^{-2} xz = 1 - t^{-2}, \qquad (4.21)$$

and

$$xt = tx, \ yt = ty, \ zt = tx,$$
$$t^{-1}t = 1 = t\ t^{-1}.$$

Thus t is a central element of T. Note that

$T = T/(t-q)T \simeq U_q(sl_2)$ and $T/(t-1)T$ is the commutative algebra $B := C[x^{\pm1}, y, z]$. By Definition 2.4, there is an induced Poisson bracket on B. In T, $[x,y] = xy-yx = t^{-2}yx-yx+(1-t^{-2}) = (1-t^{-2})(1-yx)$ therefore, in B, $\{x,y\} = 2(1-yx)$. Similarly, we obtain $\{y,z\} = 2(1-zy), \{z,x\} = 2(1-xz)$.

There are two Poisson maximal ideals of B:

(i) $I_1 = (x-1)B + (y-1)B + (z-1)B$;

(ii) $I_2 = (x+1)B + (y+1)B + (z+1)B$.

Set $u = x + y$ and $v = x + z$ and $I_1 = (u-2)B + (v-2)B + (x-1)B$. Then $x^{\pm1}, u, v$ generate B and, in terms of the new generators, the Poisson bracket becomes

$\{x,u\} = 2(1 - ux+x^2)$,

$\{x,v\} = -2(1 - vx + x^2)$,

$\{u,v\} = -2(1 - x(2v + 2u - 3x)+vu)$.

Applying the same method as before, we find that, for each $d \geq 1$, there is a d-dimensional simple Poisson B-module M, annihilated by I_1, with basis $\{m_1,\ldots,m_d\}$, such that $(x-1)M = (u-2)M = (v-2)M = 0$ and

(i) $\{x,m_j\}_M = (d - 2j + 1)m_j$ for $1 \leq j \leq d$;

(ii) $\{v,m_j\}_M = 4(j-1)(j-d-1)m_{j-1}$ for $1 \leq j \leq d$;

(iii) $\{u,m_j\}_M = m_{j+1}$ for $1 \leq j < d$ and $\{u,m_d\}_M = 0$.

In terms of x,y and z.

(i) $\{x,m_j\}_M = (\lambda - 2(j-1))m_j$ for $1 \leq j \leq d$,

(ii) $\{y,m_j\}_M = m_{j+1} - (d - 2j + 1)m_j$ for $1 \leq j < d$ and $\{y,m_d\}_M = (d-1)m_d$,

(iii) $\{z,m_j\}_M = 4(j-1)(j-d-1)m_{j-1} - (d-2j+1)m_j$, for $1 \leq j \leq d$

where $\lambda = d-1$.

To classify the finite-dimensional simple Poisson modules annihilated by I_2, we make use of the $\mathbb{C}$-algebra Poisson automorphism α of B such that

$\alpha(x) = -x$, $\alpha(y) = -y$, $\alpha(z) = -z$.

Note that $\alpha(I_1) = I_2$. The finite-dimensional simple Poisson modules annihilated by I_2 are precisely the modules M^α where M is finite-dimensional simple Poisson modules annihilated by I_1. Hence there are two d-dimensional simple Poisson modules for each $d \geq 1$.

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